On the Mutual Radiation Characteristics of Two Rigid Discs in Open or Closed Finite Circular Baffles

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ABSTRACT

Equations describing the radiation characteristics of a single rigid disc in a finite circular baffle in free space are derived using a method originally developed by Streng for a circular membrane based upon the dipole part of the Kirchhoff-Helmholtz boundary integral formula. In this case, a power series solution is derived for the radiation integral in order to avoid numerical integration. Using the product theorem, the resulting directivity function is found to be well suited for calculating the mutual radiation characteristics of two rigid discs in finite circular baffles in the same plane. The procedure is essentially the same as that used by Pritchard for calculating the mutual radiation characteristics of two rigid discs in an infinite baffle. Both are based upon Bouwkamp's method of integrating the square of directivity function, over the real and imaginary angles, in order to yield the total radiation impedance.

Finally, using an extension of Babinet's principle, it is shown how the sound radiation characteristics of a disc in a closed finite circular baffle can be obtained by combining the radiation field of a disc in an open finite circular baffle with that of a disc in an infinite baffle. A closed finite circular baffle can also be interpreted as a flanged infinite tube or a shallow cylindrical box (compared with wavelength) if the stiffness of the enclosed air is neglected. This method is also used to obtain the mutual radiation characteristics of two such sources.

1. INTRODUCTION

With the wide availability of modern finite or boundary element tools for acoustics simulation, it is perhaps tempting to neglect more traditional analytical methods. Obviously, finite element modeling is better suited to analyzing problems with complex geometries. However, if a simple geometric approximation can be made, accurate results can be obtained analytically with much less processing power than is required for a finite or boundary element analysis, especially for asymptotic expressions. For example, we may wish to calculate the characteristics of a source at a high frequency where the number of elements required would be prohibitive. The same applies to the far field response in the case of finite element modeling, although not in the case of boundary element modeling. However, if the source is not rigid, the problem becomes one of fluid-structure coupling, in which case finite elements have to be used. Other advantages to the analytical approach are

- By examining the mathematical relationships we can gain a better understanding of the physical mechanisms than when the calculations are all "hidden" in a computer.
- The equations can be used to derive design data or even lumped parameter approximations, as used by many acoustics design tools based upon circuit simulators. These enable us to establish quickly a more or less valid design prior to the FEA/BEA simulation which can then be used to fine-tune the complete design with fewer iterations. Otherwise, relying on a trial and error approach, in the case of a complex system, may not lead towards the best solution.
- It is often useful to have a benchmark against which to check the FEM/BEM simulation results. This can tell us much about the required element size and what sort of meshing geometry to use. Of course, having two ways of solving a problem gives us increased confidence in both methods.

Many of the classical analytical results, which provide the starting point for this paper, are best known through the works of Beranek [1] and Olson [2]. Fifty years ago, it must have been a formidable task reproducing these results without the benefits of modern computing power. Even today, the task would not be trivial due to the diversity of techniques employed by the authors of the original papers [3] and [4], from which these results originated. One aim of this paper is to present a simple unified approach to the problem of axially symmetric sound sources based upon the Green's function in cylindrical coordinates and the Kirchhoff-Helmholtz boundary integral formula [5]. The latter is a general equation that describes the spatial distribution of the pressure within and on the surface bounding an acoustic medium, which can be written in the general form

$$\tilde{p}(\mathbf{r}) = \iiint \tilde{f}(\mathbf{r}_0) G(\mathbf{r}|\mathbf{r}_0)\, dV_0 + \iint G(\mathbf{r}|\mathbf{r}_0) \frac{\partial}{\partial n_0} \tilde{p}(\mathbf{r}_0) - \tilde{p}(\mathbf{r}_0) \frac{\partial}{\partial n_0} G(\mathbf{r}|\mathbf{r}_0)\, dS_0 \tag{1}$$

where the first term is the volume integral over the bounded medium of the source function $f(\mathbf{r}_0)$ and the remaining two terms are the surface integrals of the boundary values of $p(\mathbf{r}_0)$ and its inward pointed normal gradient respectively over the boundary surface. $G(\mathbf{r}|\mathbf{r}_0)$ is a solution of the wave equation that satisfies the boundary conditions. If there are no boundary conditions, then $G(\mathbf{r}|\mathbf{r}_0) = g(\mathbf{r}|\mathbf{r}_0)$ which is the Green's function for an unbounded medium.

2. RADIATION CHARACTERISTICS OF A SINGLE RIGID DISC IN AN INFINITE BAFFLE

2.1. Introduction

The rigid disc in an infinite baffle is the model most commonly used for the radiation characteristics of a direct radiator loudspeaker in a sealed or vented enclosure placed in close proximity to a wall or any other kind of large flat surface. The original derivation by Lord Rayleigh [6] over one hundred years ago used an ingenious coordinate system to simplify the problem. However, in order to maintain consistency with our subsequent derivations, we shall use King's method [7] as developed by Bouwkamp [8] which uses the Green's function in cylindrical coordinates. The disc shown in Figure 1 is mounted in an infinite baffle in the xy plane and oscillates in the z direction with a harmonically time dependent velocity $\tilde{u}_0$. The area of a surface element is

$$\delta S_0 = w_0 \delta w_0 \delta \phi_0 \tag{2}$$

The monopole source elements and their images that together form the disc source are coincident in the plane of the baffle. Therefore they combine to form elements of double strength. Hence our disc in an infinite baffle can be modelled as a "breathing" disc in free space. It can also be thought of as a pulsating sphere squashed flat. Due to the symmetry of the pressure fields on either side of the line of symmetry, along the line of the infinite baffle we have the Neumann boundary condition

$$\frac{\partial}{\partial z} \tilde{p}(w,z)\Big|_{z=0+} = 0, \quad \begin{array}{c} -\infty \le w \le -a \\ a \le w \le \infty \end{array} \tag{3}$$

and on the surface of the disc we have the Neumann boundary condition

$$\frac{\partial}{\partial z} \tilde{p}(w,z)\Big|_{z=0+} = -ik\rho c \tilde{u}_0, \quad -a \le w \le -a \tag{4}$$

where ρ is the density of air or any other surrounding medium and c is the speed of sound in that medium.

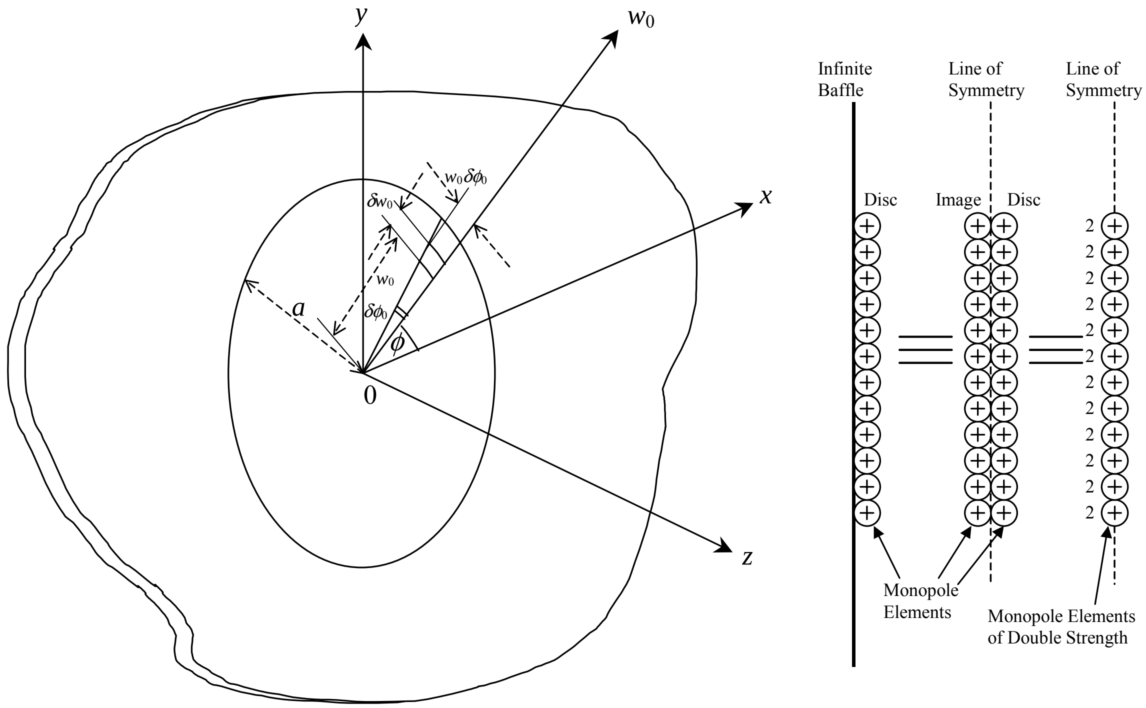

Figure 1. *Rigid circular disc in an infinite baffle*

2.2. Nearfield Pressure

The pressure distribution, in accordance with the Huygens-Fresnel principle, is given by the second term or *monopole* part of the K-H boundary integral formula (1)

$$\tilde{p}(w,z|w_0,z_0) = \int_0^{2\pi}\int_0^a g(w,z|w_0,z_0)\frac{\partial}{\partial z_0}\tilde{p}(w_0,z_0)\Big|_{z_0=0+} w_0 dw_0 d\phi_0 \tag{5}$$

where the Green's function $g(w, z\,|\,w_0, z_0)$ is the solution of the following free space wave equation in the presence of a monopole point source located at (w_0, z_0) on the surface of the diaphragm

$$\left(\frac{\partial^2}{\partial w^2}+\frac{1}{w}\frac{\partial}{\partial w}+\frac{\partial^2}{\partial z^2}+k^2\right)g(w,z|w_0,z_0) = -\delta(w-w_0,z-z_0)\begin{cases}=0,\,(w,z)\neq(w_0,z_0)\\ =\infty,\,(w,z)=(w_0,z_0)\end{cases} \tag{6}$$

where

$$k=\frac{2\pi}{\lambda}=\frac{\omega}{c}\ \text{ and }\ c=\sqrt{\frac{\gamma P_0}{\rho}} \tag{7}$$

A solution to (6) is the free space Green's function in cylindrical coordinates [5], also known as the Lamb or Sommerfeld integral, which is given by

$$g(w,\phi,z|w_0,\phi_0,z_0)=\sum_{m=0}^{\infty}\frac{i\in_m}{4\pi}\cos m(\phi-\phi_0)\int_0^{\infty}J_m(\mu w)\,J_m(\mu w_0)\frac{\mu}{\sigma}e^{-i\sigma|z-z_0|}d\mu \tag{8}$$

where

$$\sigma = \begin{cases} \sqrt{k^2 - \mu^2}, & \mu < \mathrm{k} \\ i\sqrt{\mu^2 - k^2}, & \mu > \mathrm{k} \end{cases} \tag{9}$$

and

$$\in_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases} \tag{10}$$

Due to axial symmetry, we shall only use the $m = 0$ term of the power series in (8). Inserting this together with (4) in the boundary integral (5) and integrating over the surface of the disc yields

$$\tilde{p}(w,z) = -ka\rho c\tilde{u}_0 \int_0^\infty J_1(\mu a) J_0(\mu w) \frac{e^{-i\sigma z}}{\sigma} d\mu \tag{11}$$

where we have used the identity [9]

$$\int_0^a J_0(j_{0n} w_0 / a)\, w_0 dw_0 = a^2 \int_0^1 J_0(j_{0n} t) t dt = \frac{a^2}{j_{0n}} J_1(j_{0n}), \quad \text{where } t = \frac{w_0}{a} \tag{12}$$

2.3. Radiation Impedance

We now integrate the surface pressure over the area of the disc in order to obtain the total radiation force $\tilde{F}_\mathrm{R}$ acting upon it

$$\tilde{F}_\mathrm{R} = -\int_0^{2\pi} \int_0^a \tilde{p}(w,z)\big|_{z=0+}\, w dw d\phi \tag{13}$$

After substituting (9) and (11) in (13) and evaluating the integral with $z = 0$ and again using the identity of (12) we arrive at

$$\tilde{F}_\mathrm{R} = 2\pi k a^2 \rho c \tilde{u}_0 \left[\int_0^k \frac{J_1^2(\mu a)}{\mu\sqrt{k^2 - \mu^2}} d\mu - i\int_k^\infty \frac{J_1^2(\mu a)}{\mu\sqrt{\mu^2 - k^2}} d\mu \right] \tag{14}$$

The acoustic radiation impedance z_ar is simply the ratio of the total force $\tilde{F}$ per unit area (or pressure) to the volume velocity $\tilde{U}_0$ where the volume velocity is the product of the disc surface velocity $\tilde{u}_0$ and its surface area S_D as follows

$$z_\mathrm{ar} = \frac{\tilde{F}}{S_\mathrm{D}\tilde{U}_0} = \frac{\tilde{F}}{S_\mathrm{D}^2 \tilde{u}_0} = \frac{2k\rho c}{S_\mathrm{D}} \left[\int_0^k \frac{J_1^2(\mu a)}{\mu\sqrt{k^2 - \mu^2}} d\mu - i\int_k^\infty \frac{J_1^2(\mu a)}{\mu\sqrt{\mu^2 - k^2}} d\mu \right] \tag{15}$$

where

$$S_\mathrm{D} = \pi a^2 \tag{16}$$

Using formal solutions to the real and imaginary integrals in (15) we finally obtain

$$z_\mathrm{ar} = \left(R_\mathrm{R} - iX_\mathrm{R}\right) \frac{\rho c}{S_D} \tag{17}$$

where R_R is the normalized radiation *resistance* given by

$$R_{\mathrm{R}} = 1 - \frac{J_1(2ka)}{ka} \tag{18}$$

and X_{R} is the normalized radiation *reactance* given by

$$X_{\mathrm{R}} = \frac{\mathbf{H}_1(2ka)}{ka} \tag{19}$$

where J_1 is the 1st order Bessel function and $\mathbf{H}_1$ is the 1st order Struve function.

2.4. Far Field Response

In the case of the far field response, it is more convenient to use polar coordinates so that we can obtain the far field polar responses directly. Rayleigh's far field approximation [5] is ideal for this purpose

$$g(r,\vartheta,\phi|w_0,\phi_0,z_0) = \frac{e^{-ik\left(r - w_0 \sin\vartheta \cos(\phi-\phi_0) - z_0 \cos\vartheta\right)}}{4\pi r} \tag{20}$$

Again we shall use the second term or *monopole* part of the K-H boundary integral formula (1)

$$\tilde{p}(r,\vartheta,\phi) = \int_0^{2\pi}\int_0^a g(r,\vartheta,\phi|w_0,\phi_0,z_0)\Big|_{z_0=0+} \frac{\partial}{\partial z_0}\tilde{p}(w_0,\phi_0,z_0)\Big|_{z_0=0+} w_0 dw_0 d\phi_0 \tag{21}$$

As the disc is axially symmetric, we can choose any reference angle ϕ in (20). Therefore we can simplify the problem by letting $\phi = \pi/2$ so that $\cos(\phi - \phi_0) = \sin\phi_0$. We can now insert (20) and (4) into the K-H integral (21) and integrate over the surface area of the disc to obtain

$$\tilde{p}(r,\vartheta) = -ik\rho c S_{\mathrm{D}} \tilde{u}_0 \frac{e^{-ikr}}{2\pi r} D(\vartheta) \tag{22}$$

where $S_{\mathrm{D}} = \pi a^2$ is the area of the disc and $D(\vartheta)$ is the directivity function given by

$$D(\vartheta) = \frac{2J_1(ka\sin\vartheta)}{ka\sin\vartheta} \tag{23}$$

and we have used the identities [9]

$$\frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\phi_0}\, d\phi_0 = J_0(t), \quad \text{where}\, t = kw_0 \sin\vartheta \tag{24}$$

and (12) with $j_{0n}/a = k\sin\vartheta$. In order to evaluate the on axis pressure, we simply set $\vartheta = 0$ in equation (22). The result is most commonly written

$$\tilde{p}(r,0) = -i\omega\rho\tilde{U}_0 \frac{e^{-ikr}}{2\pi r} \tag{25}$$

where the volume velocity is given by $\tilde{U}_0 = S_{\mathrm{D}}\tilde{u}_0$.

3. RADIATION CHARACTERISTICS OF A SINGLE RIGID DISC IN AN OPEN FINITE CIRCULAR BAFFLE IN FREE SPACE

3.1. Introduction

It is not always appropriate to mount a loudspeaker in an enclosure. For example, if the enclosure has to be so small that it shifts the main resonant frequency up an octave or two, the bass response will be severely limited. In the case of speakers with large lightweight diaphragms such as electrostatics, even a moderate size box has a severe impact on the bass response. In such cases, it may be more appropriate to leave the speaker open at the back. The cost of this, though, is that sound from the back at low frequencies cancels that from the front. However, in some cases this is less severe than the loss due to an enclosure of less than optimum size. Nimura and Watanabe [10] calculated the radiation characteristics of a disc in a circular baffle for values of *ka* up to 2. However, here we shall adapt the method used by Streng [11] & [12] for an unbaffled circular membrane. The latter provides a far field expression which is readily suitable for calculating the mutual radiation impedance of two discs in finite circular baffles or free space using the same method developed by Pritchard [13] for calculating the mutual radiation impedance of two rigid circular discs in an infinite baffle. The disc shown in Figure 2 lies in the xy plane and oscillates in the *z* direction with velocity $\tilde{u}_0$, thus radiating sound from both sides. It has a radius *a* and is assumed to be infinitesimally thin. The inner and outer radii of the stationary baffle are *a* and *b* respectively. If $b = a$, then the problem reduces to that of a *rigid disc in free space*. The area of a surface element is given by (2). We would expect the pressure field on one side of the *xy* plane to be the symmetrical "negative" of that on the other, so that

$$\begin{aligned} p(w,z) &= -p(w,-z) \\ p(w,0) &= 0, \quad w > a \end{aligned} \tag{26}$$

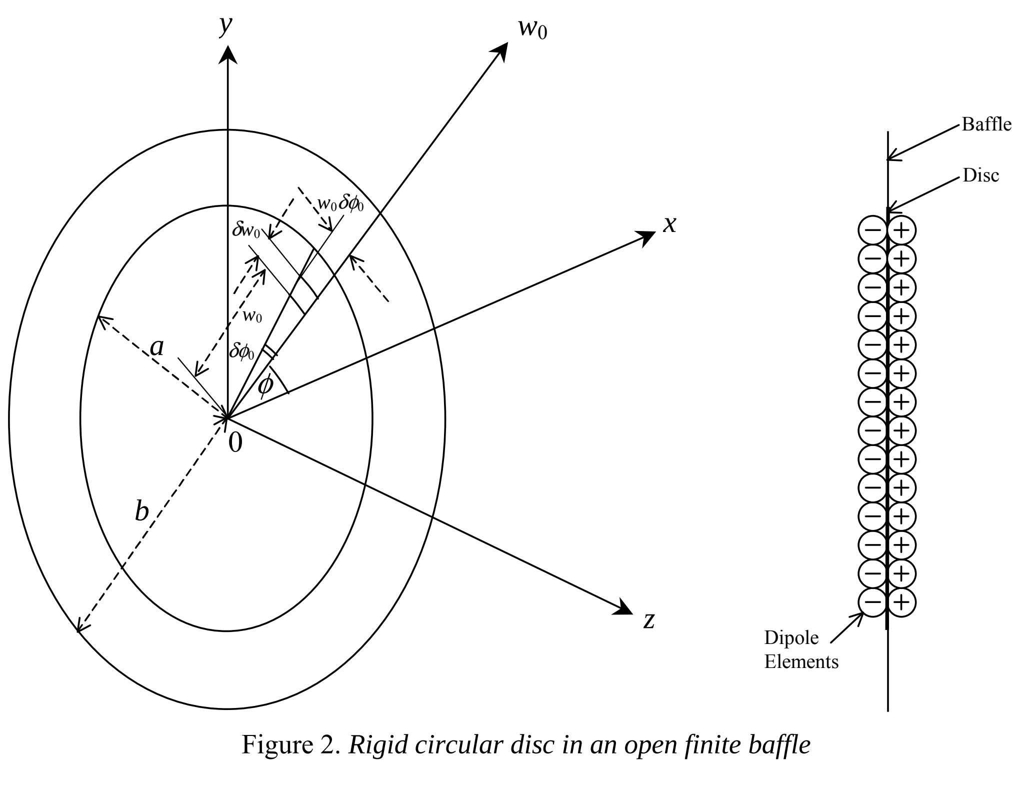

Figure 2. *Rigid circular disc in an open finite baffle*

Also, at the front and rear outer surfaces of the disc, we have the coupling condition

$$\frac{\partial}{\partial z}\tilde{p}(w,z)\Big|_{z=0\pm} = -ik\rho c\tilde{u}_0(w), \quad 0 \le w \le a \tag{27}$$

and at the front and rear outer surfaces of the baffle, we have the boundary condition

$$\frac{\partial}{\partial z}\tilde{p}(w,z)\Big|_{z=0\pm} = 0, \quad a \le w \le b \tag{28}$$

where ρ is the density of air or any other surrounding medium and c is the speed of sound in that medium. In order to tackle this problem, we shall use the dipole surface integral part of the K-H equation (1). However, this differs from the monopole part previously used for the disc in an infinite baffle in that we now need some prior expression for the frontal surface pressure distribution $\tilde{p}_+(w_0)$. Also, because the disc can radiate from both sides, we must include the rear surface pressure distribution $\tilde{p}_-(w_0)$ too, where $\tilde{p}_+(w_0) = -\tilde{p}_-(w_0)$. Streng [11] & [12] showed that the surface pressure distribution for any flat axially-symmetric unbaffled source (or sink), based upon Bouwkamp's solution [14] to the free space wave equation in oblate spheroidal coordinates, could be written as

$$\tilde{p}_+(w_0) = -\tilde{p}_-(w_0) = \sum_{m=0}^{\infty}\tilde{A}_m\left(1-\frac{w_0^2}{a^2}\right)^{m+\frac{1}{2}} \tag{29}$$

In some literature [5], this pressure is taken as approximately constant across the radius.

3.2. Solution of the Free Space Wave Equation

The pressure distribution, in accordance with the Huygens-Fresnel principle, is given by the *dipole* part of the Kirchhoff-Helmholtz boundary integral formula

$$\tilde{p}(w,z|w_0,z_0) = -\int_0^{2\pi}\int_0^b\left(\tilde{p}_+(w_0)-\tilde{p}_-(w_0)\right)\frac{\partial}{\partial z_0}g(w,z|w_0,z_0)\Big|_{z_0=0+}w_0\,dw_0\,d\phi_0 \tag{30}$$

where the Green's function $g(w\,|\,w_0)$ is the solution of the free space wave equation (6) in the presence of a monopole point source located at (w_0, z_0) on the surface of the disc. However, the pressure produced at each point (w,z) in space by each dipole element is defined in the integral (30) by the product of the surface pressure, the inward pointed normal gradient of the Green's function and the area of each element given by (2). A solution to (6) is given by the Green's function of (8). However, we need to evaluate its normal gradient at the surface as follows

$$\frac{\partial}{\partial z_0}g(w,z|w_0,z_0)\Big|_{z_0=0+} = \sum_{m=0}^{\infty}\frac{\in_m}{4\pi}\cos m(\phi-\phi_0)\int_0^{\infty}J_m(\mu w)\,J_m(\mu w_0)\mu e^{-i\sigma z}d\mu \tag{31}$$

Due to axial symmetry, we shall only use the $m = 0$ term of the power series in (31). Inserting the power series (29) and (31) in the boundary integral (30) and integrating over the surface of the disc and its baffle yields

$$\tilde{p}(w,z) = -b\sum_{m=1}^{\infty}\tilde{A}_m 2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})\int_0^{\infty}\left(\frac{1}{\mu b}\right)^{m+\frac{1}{2}}J_{m+\frac{3}{2}}(\mu b)J_0(\mu w)e^{-i\sigma z}d\mu \tag{32}$$

where we have used the identity

$$\int_0^b w_0\left(1-\frac{w_0^2}{b^2}\right)^{m+\frac{1}{2}} J_0(\mu w_0)dw_0$$
$$= b^2\int_0^1 t\left(1-t^2\right)^{m+\frac{1}{2}} J_0(\mu bt)dt = b^2 2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})\left(\frac{1}{\mu b}\right)^{m+\frac{3}{2}} J_{m+\frac{3}{2}}(\mu b), \quad \text{where } t=\frac{w_0}{b} \tag{33}$$

If we now apply the boundary conditions of (27) and (28), we obtain

$$\frac{\partial}{\partial z}\tilde{p}(w,z)\Big|_{z=0\pm} = ib\sum_{m=1}^{\infty}\tilde{A}_m 2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})\int_0^{\infty}\left(\frac{1}{\mu b}\right)^{m+\frac{1}{2}} J_{m+\frac{3}{2}}(\mu b)J_0(\mu w)\sigma d\mu$$
$$= \begin{cases} -ik\rho c\tilde{u}_0, & 0\le w\le a \\ 0, & a\le w\le b \end{cases} \tag{34}$$

Equation (34) can be written more simply as

$$\sum_{m=0}^{\infty}\tau_m I_m = \begin{cases} -1, & 0\le w\le a \\ 0, & a\le w\le b \end{cases} \tag{35}$$

where

$$\tau_m = \frac{2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})\tilde{A}_m}{kb\rho c\tilde{u}_0} \tag{36}$$

and

$$I_m(w,k) = I_{mR}(w,k) + iI_{mI}(w,k) \tag{37}$$

where the real part of the integral (37) is given by

$$I_{mR}(w,k) = b^2\int_0^k\left(\frac{1}{\mu b}\right)^{m+\frac{1}{2}}\sqrt{k^2-\mu^2}\; J_{m+\frac{3}{2}}(\mu b)J_0(\mu w)d\mu \tag{38}$$

and the imaginary part is given by

$$I_{mI}(w,k) = b^2\int_k^{\infty}\left(\frac{1}{\mu b}\right)^{m+\frac{1}{2}}\sqrt{\mu^2-k^2}\; J_{m+\frac{3}{2}}(\mu b)J_0(\mu w)d\mu \tag{39}$$

3.3. Solution of the Real Integral

In order to solve the real integral (38) we will substitute $\mu = k\sin\theta$ so that

$$I_{mR}(w,k) = \frac{1}{(kb)^{m-3/2}}\int_0^{\pi/2}\frac{J_{m+3/2}(kb\sin\theta)}{\sin^{m+3/2}\theta}J_0(kw\sin\theta)\cos^2\theta\sin\theta\, d\theta \tag{40}$$

Using the same method as Pritchard [13], we can apply a Lommel expansion to the first term of the integral (40) as follows

$$\frac{J_{m+3/2}(kb\sin\theta)}{\sin^{m+3/2}\theta} = \frac{J_{m+3/2}\left(kb\sqrt{1-\cos^2\theta}\right)}{(1-\cos^2\theta)^{m/2+3/4}} = \sum_{q=0}^{Q}\frac{(kb)^q\cos^{2q}\theta}{2^q q!}J_{m+q+\frac{3}{2}}(kb) \tag{41}$$

Inserting (41) in (40) yields

$$I_{mR}(w,k) = \frac{1}{(kb)^{m-3/2}}\sum_{q=0}^{Q}\frac{(kb)^q}{2^q q!}J_{m+q+\frac{3}{2}}(kb)\int_0^{\pi/2}J_0(kw\sin\theta)\cos^{2(q+1)}\theta\sin\theta\, d\theta \tag{42}$$

After evaluating the integral over θ we arrive at

$$I_{mR}(w,k) = \sqrt{2}\sum_{q=0}^{Q}\frac{\Gamma(q+3/2)}{q!(kb)^m(w/b)^{q+3/2}}J_{m+q+\frac{3}{2}}(kb)J_{q+\frac{3}{2}}(kw) \tag{43}$$

where we have used Sonine's integral [9] as follows

$$\int_0^{\pi/2}J_0(kw\sin\theta)\cos^{2(q+1)}\theta\sin\theta\, d\theta = \sum_{q=0}^{Q}\frac{2^{q+1/2}\Gamma(q+3/2)}{(kw)^{q+3/2}}J_{q+\frac{3}{2}}(kw) \tag{44}$$

3.4. Solution of the Imaginary Integral

Applying the Lommel expansion in a similar way to the imaginary integral (39) yields exactly the same expression as (43) except that the Bessel function $J_{q+3/2}$ is now replaced with the Neumann function $Y_{q+3/2}$ of the same order. Unfortunately, the series does not converge for $w < a$, which rules out its use for calculating the power series coefficients or self radiation impedance. However, we will use this method for calculating the mutual radiation impedance of two rigid discs in which case the series does converge if the distance between the centers of the discs is greater than the sum of their radii. Streng [11] showed that by replacing the Bessel function $J_{q+3/2}$ with Hankel functions $H^{(1)}_{q+3/2} + H^{(2)}_{q+3/2}$, whilst applying contour integration theory and substituting $\mu = ke^{i\theta}$, the imaginary integral (39) could be expressed as

$$I_{mI}(w,k) = -\Re e\left(\frac{i}{(kb)^{m-3/2}}\int_0^{\pi/2}e^{i\left(\frac{1}{2}-m\right)\theta}\sqrt{e^{2i\theta}-1}\,J_0(kwe^{i\theta})\left[J_{m+\frac{3}{2}}(kbe^{i\theta}) + iY_{m+\frac{3}{2}}(kbe^{i\theta})\right]d\theta\right) \tag{45}$$

The integral of (45) has finite limits and is therefore easier to calculate numerically than the infinite integral of (39), However, the integrand is still strongly oscillatory which can lead to significant loss of precision. We can expand the Bessel and Neumann functions [9] as follows

$$J_0(kwe^{i\theta}) = \sum_{q=0}^{Q}\left(\frac{kw}{2}\right)^{2q}\frac{(-1)^q e^{2i\theta q}}{(q!)^2} \tag{46}$$

$$J_{m+\frac{3}{2}}(kbe^{i\theta}) = \sum_{q=0}^{Q}\left(\frac{kb}{2}\right)^{2q+m+\frac{3}{2}}\frac{(-1)^q e^{i\theta(2q+m+3/2)}}{q!\Gamma(q+m+5/2)} \tag{47}$$

$$Y_{m+\frac{3}{2}}(kbe^{i\theta}) = \sum_{q=0}^{Q} \left(\frac{kb}{2}\right)^{2q-m-\frac{3}{2}} \frac{(-1)^{q+m} e^{i\theta(2q-m-3/2)}}{q!\Gamma(q-m-1/2)} \tag{48}$$

After substituting (46), (47) and (48) in (45) we have

$$I_{mI}(w,k) = -2^{\frac{3}{2}-m} \Re e \sum_{q=0}^{Q}\sum_{r=0}^{R} \left(\begin{array}{l} \left(\frac{kb}{2}\right)^{2(q+r)+3} \left(\frac{w}{b}\right)^{2q} \dfrac{(-1)^{q+r}}{(q!)^2 r!\Gamma(r+m+5/2)} i\int_0^{\pi/2} \sqrt{e^{2i\theta}-1}\, e^{2i\theta(q+r+1)} d\theta \\ +i\left(\frac{kb}{2}\right)^{2(q+r-m)} \left(\frac{w}{b}\right)^{2q} \dfrac{(-1)^{q+r+m}}{(q!)^2 r!\Gamma(r-m-1/2)} i\int_0^{\pi/2} \sqrt{e^{2i\theta}-1}\, e^{2i\theta(q+r-m-1/2)} d\theta \end{array} \right) \tag{49}$$

In order to solve the integrals in (49), we can use the identity

$$\int_0^{2\pi} \sqrt{e^{2i\theta}-1}\, e^{2i\beta\theta} d\theta = \frac{1}{2\beta}\left[{}_2F_1\left(-\frac{1}{2}, \beta; \beta+1; -1\right) e^{i\pi\beta} - \frac{\sqrt{\pi}\,\Gamma(\beta+1)}{2\,\Gamma(\beta+3/2)} \right] \tag{50}$$

After evaluating the integrals over θ we arrive at

$$I_{mI}(w,k) = 2^{\frac{3}{2}-m} \Re e \sum_{q=0}^{Q}\sum_{r=0}^{R} \left(F_Y(q,r,m)\left(\frac{kb}{2}\right)^{2(q+r-m)} \left(\frac{w}{b}\right)^{2q} - iF_J(q,r,m)\left(\frac{kb}{2}\right)^{2(q+r)+3} \left(\frac{w}{b}\right)^{2q} \right) \tag{51}$$

where

$$F_Y(q,r,m) = (-1)^{q+r+m} \frac{ {}_2F_1\left(-\frac{1}{2}, q+r-m-\frac{1}{2}; q+r-m+\frac{1}{2}; -1\right) e^{i\pi(q+r-m-1/2)} - \dfrac{\sqrt{\pi}\,\Gamma(q+r-m+1/2)}{2\,\Gamma(q+r-m+1)} }{2\,(q+r-m-1/2)\,(q!)^2 r!\,\Gamma(r-m-1/2)} \tag{52}$$

$$F_J(q,r,m) = (-1)^{q+r} \frac{ {}_2F_1\left(-\frac{1}{2}, q+r+1; q+r+2; -1\right) e^{i\pi(q+r+1)} - \dfrac{\sqrt{\pi}\,\Gamma(q+r+2)}{2\,\Gamma(q+r+5/2)} }{2\,(q+r+1)\,(q!)^2 r!\,\Gamma(r+m+5/2)} \tag{53}$$

However, for integer values of q and r, $iF_J(q,r,m)$ is purely imaginary and therefore makes no contribution to the real part of $I_{mI}(w,k)$. Similarly, the $e^{i\pi(q+r-m-1/2)}$ term of $F_Y(q,r,m)$ is also purely imaginary and can therefore be excluded. Thus the final result can be written

$$I_{mI}(w,k) = -\frac{\sqrt{\pi}}{2^{m+1/2}} \sum_{q=0}^{Q}\sum_{r=0}^{R} \frac{(-1)^{q+r+m}\Gamma(q+r-m-1/2)}{(q!)^2 r!\,\Gamma(r-m-1/2)\,\Gamma(q+r-m+1)} \left(\frac{kb}{2}\right)^{2(q+r-m)} \left(\frac{w}{b}\right)^{2q} \tag{54}$$

3.5. Calculation of the Power Series Coefficients

Equation (35) can be solved using the *least mean squares* or LMS algorithm. Firstly, we shall define an error function

$$E(\tau_m) = \sum_{p=1}^{P} \left| \sum_{m=0}^{M} \tau_m(k) I_m(w_p,k) + \in(w_p) \right|^2 \tag{55}$$

where

$$w = w_p = \frac{p}{P} b \text{ , and } \in (w_p) = \begin{cases} 1, & 0 \le w_p \le a \\ 0, & a \le w_p \le b \end{cases} \tag{56}$$

In order to find the minimum value of $E(\tau_m)$ we shall equate its derivative to zero and solve for τ_m as follows

$$\frac{d}{d\tau_m} E(\tau_m) = 2\sum_{p=1}^{P} I_j(w_p, k) \left[\sum_{m=0}^{M} \tau_m I_m(w_p, k) + \in (w_p) \right] = 0 \tag{57}$$

for j = 1, 2,⋯,M. We now have to solve the following set of M simultaneous equations for τ_m

$$\sum_{m=0}^{M} \tau_m \sum_{p=1}^{P} I_j(w_p, k) I_m(w_p, k) = -\sum_{p=1}^{aP/b} I_j(w_p, k) \tag{58}$$

3.6. Surface Pressure

Using the expression for τ_m in (36), we can write

$$\widetilde{A}_m = \frac{\tau_m k b \rho c \widetilde{u}_0}{2^{m+\frac{1}{2}} \Gamma(m + \frac{3}{2})} \tag{59}$$

After substituting this in (29), the surface pressure can be written as

$$\widetilde{p}_+(w_0) = k b \rho c \widetilde{u}_0 \sum_{m=0}^{M} \frac{\tau_m}{2^{m+\frac{1}{2}} \Gamma(m + \frac{3}{2})} \left(1 - \frac{w_0^2}{b^2} \right)^{m+\frac{1}{2}} \tag{60}$$

3.7. Radiation Impedance

In order to find the total force $\widetilde{F}$ acting upon the disc, we integrate the pressure from (60) over the whole surface on *both* sides as follows

$$\begin{aligned} \widetilde{F} &= -2 k b \rho c \widetilde{u}_0 \int_0^{2\pi} \int_0^a \widetilde{p}_+(w_0) w_0 dw_0 d\phi_0 \\ &= -2\pi k b^3 \rho c \widetilde{u}_0 \sum_{m=0}^{M} \frac{\tau_m}{2^{m+\frac{1}{2}} \Gamma(m + \frac{5}{2})} \left[1 - \left(1 - \frac{a^2}{b^2} \right)^{m+\frac{3}{2}} \right] \end{aligned} \tag{61}$$

The acoustic radiation impedance z_{ar} is then given by

$$z_{ar} = \frac{\widetilde{F}}{S_D \widetilde{U}_0} = \frac{\widetilde{F}}{S_D^2 \widetilde{u}_0} = -\frac{2 k b \rho c}{\pi a^2} \left(\frac{b}{a} \right)^2 \sum_{m=0}^{M} \frac{\tau_m}{2^{m+\frac{1}{2}} \Gamma(m + \frac{5}{2})} \left[1 - \left(1 - \frac{a^2}{b^2} \right)^{m+\frac{3}{2}} \right] \tag{62}$$

where $\widetilde{U}_0$ is the total volume velocity produced by the disc which is the product of its surface velocity $\widetilde{u}_0$ and its area S_D where $S_D = \pi a^2$. Finally, we can write the radiation impedance as

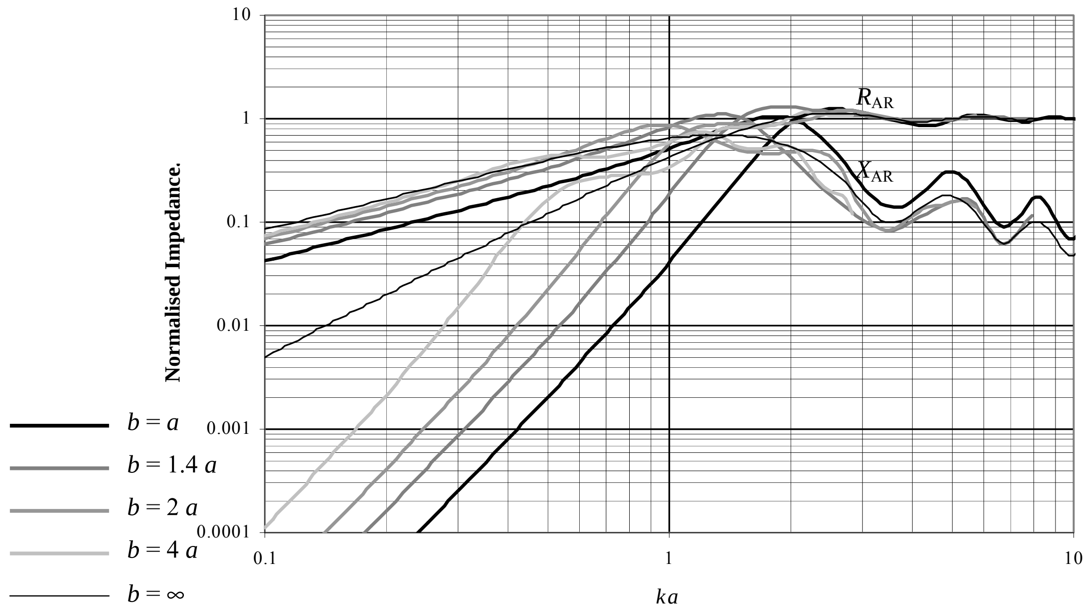

Figure 3. *Normalized radiation impedance of a rigid disc in an open circular baffle*

$$z_{\text{ar}} = \frac{2\rho c}{S_D}\left(R_{\text{R}} + iX_{\text{R}}\right) \tag{63}$$

where R_{R} is the normalized radiation *resistance* given by

$$R_{\text{R}} = -ka\left(\frac{b}{a}\right)^3 \sum_{m=0}^{M} \frac{\Re e(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})}\left[1-\left(1-\frac{a^2}{b^2}\right)^{m+\frac{3}{2}}\right] \tag{64}$$

and X_{R} is the normalized radiation *reactance* given by

$$X_{\text{R}} = -ka\left(\frac{b}{a}\right)^3 \sum_{m=0}^{M} \frac{\Im m(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})}\left[1-\left(1-\frac{a^2}{b^2}\right)^{m+\frac{3}{2}}\right] \tag{65}$$

This result is plotted in Figure 3 for various ratios of *b* to *a*.

3.8. Far-Field Response

Again, we shall use the K-H integral (30), but first we need to find the surface normal gradient of the far field Green's function given in (20)

$$\frac{\partial}{\partial z_0} g(r,\vartheta,\phi|w_0,\phi_0,z_0)\Big|_{z_0=0+} = ik\cos\vartheta \frac{e^{-ikr}}{4\pi r} e^{ikw_0 \sin\vartheta \cos(\phi-\phi_0)} \tag{66}$$

Again, we let $\phi = \pi/2$ and insert (66) together with the power series (60) into the K-H integral (30) and integrate over the surface of the disc and baffle to obtain

$$\tilde{p}(r,\vartheta) = -ik^2 b^3 \rho c \tilde{u}_0 \cos\vartheta \frac{e^{-ikr}}{r} \sum_{m=0}^{M} \tau_m \left(\frac{1}{kb\sin\vartheta}\right)^{m+\frac{3}{2}} J_{m+\frac{3}{2}}(kb\sin\vartheta) \tag{67}$$

where we have used the identities (12) and (33) with $\mu = k \sin \vartheta$. We can now write the complete expression for the far field pressure in the form

$$\tilde{p}(r,\vartheta) = -ik\rho c S_{\mathrm{D}} \tilde{u}_0 \frac{e^{-ikr}}{2\pi r} D(\vartheta) \tag{68}$$

where $S_{\mathrm{D}} = \pi a^2$ is the area of the disc and $D(\vartheta)$ is the directivity function given by

$$D(\vartheta) = -2kb\left(\frac{b}{a}\right)^2 \cos\vartheta \sum_{m=0}^{M} \tau_m \frac{J_{m+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{m+3/2}} \tag{69}$$

In order to evaluate the on axis pressure, we simply set $\vartheta = 0$ in equation (66) before inserting it in (30). This gives us an integral that is the same as the one for the radiation impedance in equation (61) except that the pressure is integrated over the whole surface of the disc together with the baffle rather than just the disc itself. Hence

$$D(0) = -kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\tau_m}{2^{m+\frac{1}{2}} \Gamma(m+\frac{5}{2})} \tag{70}$$

The normalized on-axis responses for various ratios of b/a are shown in Figure 4 where the normalized SPL is given by

$$SPL_{\mathrm{Norm}} = 20 \log_{10} D(0) \tag{71}$$

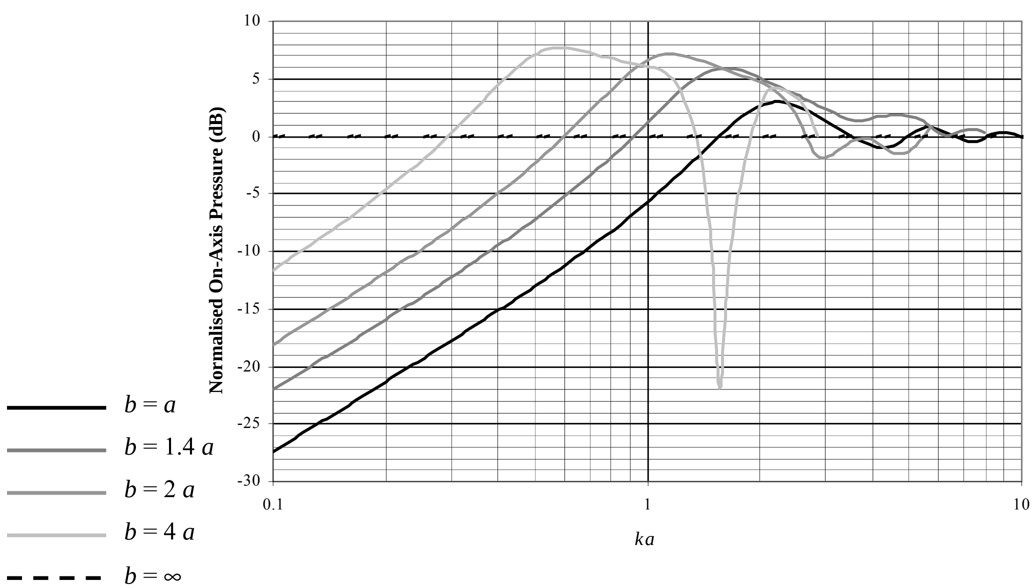

Figure 4. *Normalized on-axis response of a rigid disc in an open circular baffle*

4. RADIATION CHARACTERISTICS OF A SINGLE RIGID DISC IN A CLOSED FINITE CIRCULAR BAFFLE IN FREE SPACE

4.1. Introduction

The disc shown in Figure 5 lies in the xy plane and oscillates in the z direction with velocity $\tilde{u}_0$, radiating sound from the front side only. The sound from its rear is blocked by the cylindrical box, which has a radius *b* and a depth *h*. The disc has a radius *a*

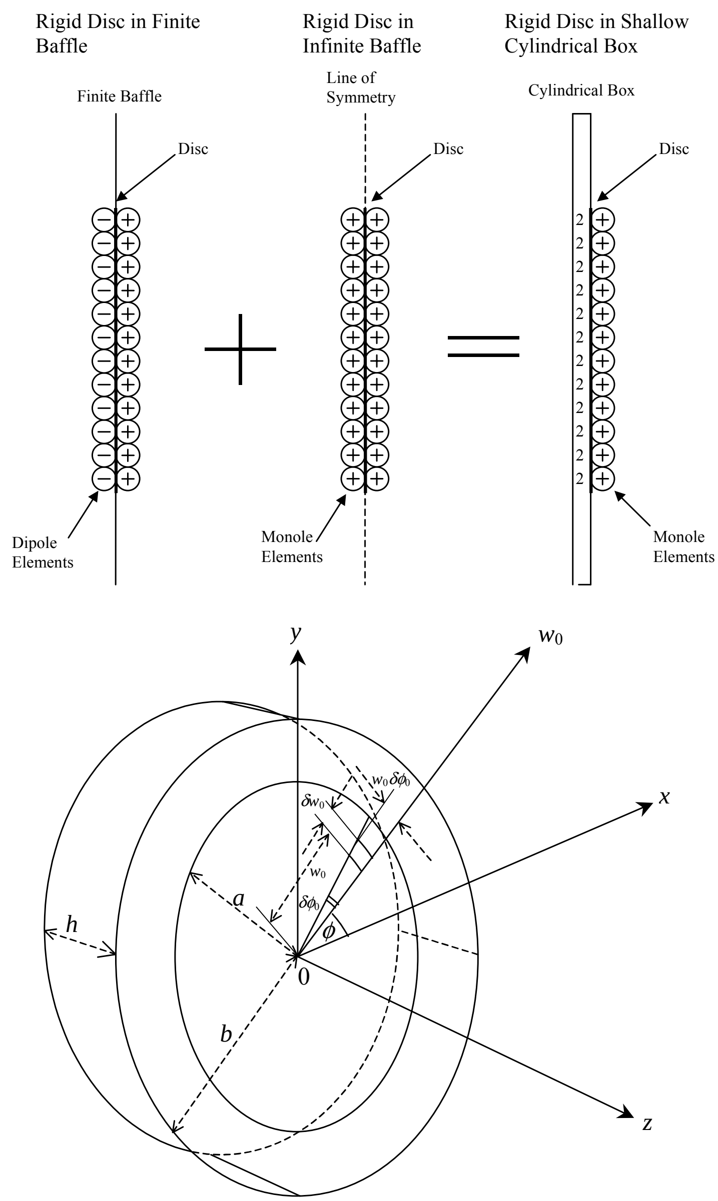

Figure 5. *Rigid circular disc in an closed finite baffle* (*or shallow cylindrical box*)

The model is valid providing $h \le a/4$ and may also be considered as a *rigid disc at the end of a flanged infinite tube*, where b is the outer radius of the flange. If $b = a$, then the problem reduces to that of a *rigid disc at the end of an unflanged infinite tube*. The area of a surface element is given by (2).

4.2. Radiation Impedance

From Figure 1, we see that the front surface pressure is the sum of the surface pressures of both a disc in an infinite baffle and a disc in a finite baffle. However, the resulting pressure is double the strength of each, so we must divide the result by two. We can apply this argument to the radiation impedance too, which is simply proportional to the sum of the surface pressures. The real part of the normalized radiation impedance can thus be obtained by combining (18) and (64) as follows

$$R_{\mathrm{R}} = \frac{1}{2}\left\{1 - \frac{J_1(2ka)}{ka} - kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\Re e(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})}\left[1-\left(1-\frac{a^2}{b^2}\right)^{m+\frac{3}{2}}\right]\right\} \tag{72}$$

Similarly, the imaginary part can be obtained by combining (19) and (65)

$$X_{\mathrm{R}} = \frac{1}{2}\left\{\frac{H_1(2ka)}{ka} - kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\Im m(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})}\left[1-\left(1-\frac{a^2}{b^2}\right)^{m+\frac{3}{2}}\right]\right\} \tag{73}$$

This result is plotted in Figure 6 for various ratios of b to a.

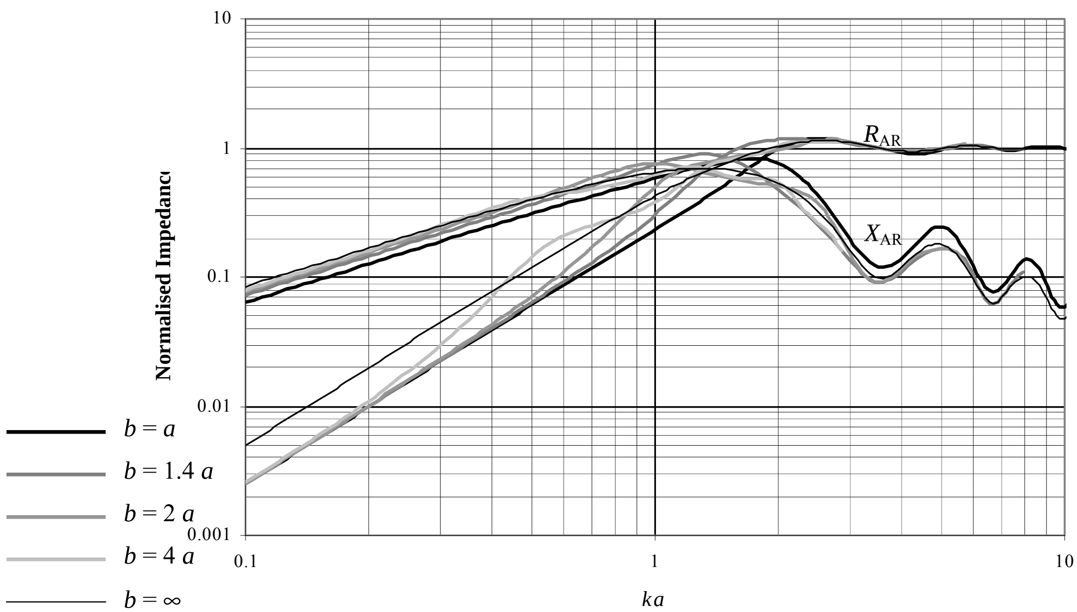

Figure 6. *Normalized radiation impedance of a rigid disc in a closed circular baffle*

4.3. Far Field Response

The far field response takes on the same form as that for an infinite baffle (22) and an open baffle (68)

$$\tilde{p}(r,\vartheta) = -ik\rho c S_\mathrm{D}\tilde{u}_0 \frac{e^{-ikr}}{2\pi r} D(\vartheta) \tag{74}$$

where $S_\mathrm{D} = \pi a^2$ is the area of the disc and $D(\vartheta)$ is the directivity function obtained by combining that of an infinite baffle (23) with an open baffle (69)

$$D(\vartheta) = \frac{J_1(ka\sin\vartheta)}{ka\sin\vartheta} - kb\left(\frac{b}{a}\right)^2 \cos\vartheta \sum_{m=0}^{M} \tau_m \frac{J_{m+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{m+3/2}} \tag{75}$$

Similarly, the on-axis response may be obtained by combining the on-axis response of an infinite baffle, which is just unity, with that of an open baffle (70)

$$D(0) = \frac{1}{2}\left\{1 - kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\tau_m}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})}\right\} \tag{76}$$

The normalized on-axis responses for various ratios of b/a are shown in Figure 7 again using (71).

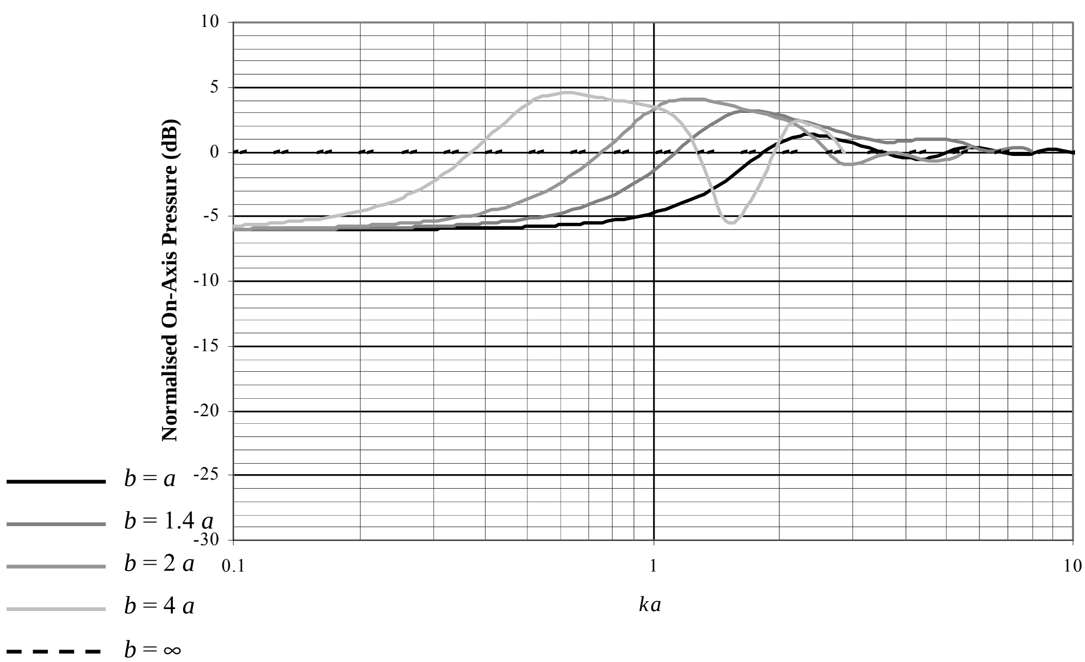

Figure 7. *Normalized on-axis response of a rigid disc in a closed circular baffle*

5. Bouwkamp's Impedance Theorem

It is perhaps worth giving a brief summary of this important theorem [8]. The real power N radiated from a circular disc of radius a can be expressed as the product of the square of its volume velocity $\widetilde{U}_0$ and the radiation resistance r_{ar}.

$$N = \widetilde{U}_0^2 r_{\mathrm{ar}} = (S_{\mathrm{D}} \widetilde{u}_0)^2 r_{\mathrm{ar}} \tag{77}$$

The real power can also be calculated from the mean square of the far field pressure $\widetilde{p}(r, \vartheta)$ over a hemispherical surface at a distance r divided by the far field radiation impedance $z_0(r)$ as r tends to infinity

$$N = \frac{1}{z_0(r)} \cdot \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \widetilde{p}^2(r, \vartheta) \sin \vartheta \, d\vartheta \, d\phi \tag{78}$$

where

$$z_0(r) = \frac{\rho c}{2\pi r^2} \tag{79}$$

If we insert the far field pressure for a disc in an infinite baffle (22) and (79) in (78) and equate this with (77), we obtain

$$r_{\mathrm{ar}} = \frac{\rho c k^2}{4\pi^2} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} D^2(\vartheta) \sin \vartheta \, d\vartheta \, d\phi \tag{80}$$

More generally, we can write

$$z_{\mathrm{ar}} = \frac{\rho c}{S_{\mathrm{D}}} Z_{\mathrm{R}} = \frac{\rho c}{S_{\mathrm{D}}} (R_{\mathrm{R}} + iX_{\mathrm{R}}) \tag{81}$$

where

$$R_{\mathrm{R}} = \frac{(ka)^2}{4\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} D^2(\vartheta) \sin \vartheta \, d\vartheta \, d\phi \tag{82}$$

and

$$X_{\mathrm{R}} = \frac{(ka)^2}{4\pi} \int_0^{2\pi} \int_{\frac{\pi}{2}+i0}^{\frac{\pi}{2}+i\infty} D^2(\vartheta) \sin \vartheta \, d\vartheta \, d\phi \tag{83}$$

It is fairly straightforward to verify this result by substituting the directivity function (23) and $\mu = k \sin \vartheta$ in (82) and (83). In this way, the expressions (18) and (19) can be duplicated, bearing in mind that $\sin(\pi/2+i\infty) = \cos i\infty = \cosh \infty = \infty$. Of course, this theorem is not limited to radiators with uniform surface velocity. Bouwkamp's expression includes the square of average surface velocity divided by the square of the velocity at some reference point, although we have omitted this here.

6. MUTUAL RADIATION CHARACTERISTICS OF TWO RIGID DISCS IN OPEN FINITE CIRCULAR BAFFLES IN FREE SPACE

6.1. Introduction

The two discs shown in Figure 8 both lie in the xy plane and oscillate in the z direction with velocity $\tilde{u}_0$, thus radiating sound from both sides. Each has a radius a and is assumed to be infinitesimally thin. The inner and outer radii of the stationary baffles are a and b respectively. If $b = a$, then the problem reduces to that of *two rigid discs in free space*.

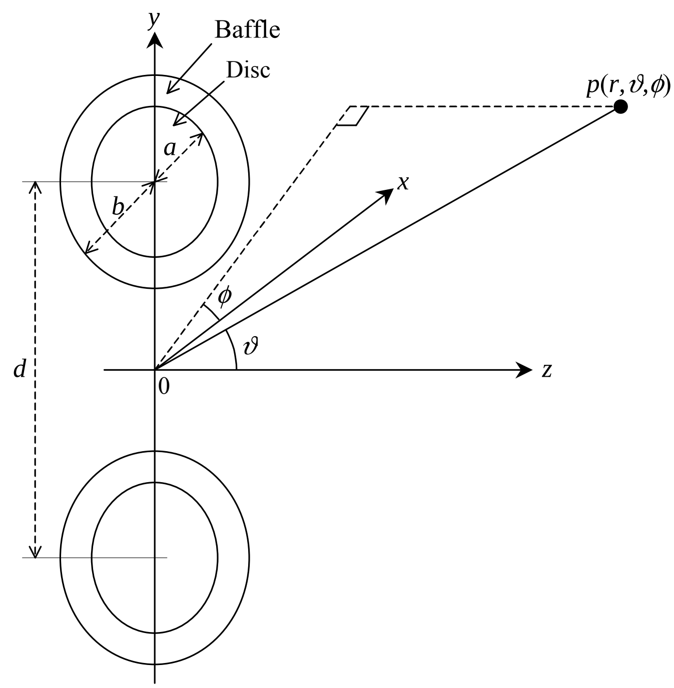

Figure 8. *Two rigid circular discs in open finite circular baffles*

6.2. Directional Response

Using the product theorem, the directivity function D of the over-all radiator comprising the two identical discs, as shown in Figure 10, is simply the product of the directivity function of a single radiator (69) with the directivity function of two point elements

$$D(\vartheta,\phi) = -2kb\left(\frac{b}{a}\right)^2 \cos\vartheta \sum_{m=0}^{M} \tau_m \frac{J_{m+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{m+3/2}} \cdot 2\cos\left(\frac{kd}{2}\sin\vartheta\sin\phi\right) \tag{84}$$

By making use of the relation $\cos\alpha = \sqrt{(1+\cos 2\alpha)/2}$ together with (81), (82) and (83) we obtain

$$Z_{\mathrm{R}} = \frac{-(kb)^4}{\pi}\left(\frac{b}{a}\right)^2 \int_0^{2\pi}\int_0^{\frac{\pi}{2}+i\infty} \sum_{m=0}^{M}\sum_{j=0}^{M} \tau_m \frac{J_{m+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{m+3/2}} \tau_j \frac{J_{j+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{j+3/2}} \left[1+\cos(kd\sin\vartheta\sin\phi)\right]\cos^2\vartheta\sin\vartheta\, d\vartheta\, d\phi \tag{85}$$

where we have used the shorthand

$$\int_0^{\frac{\pi}{2}+i\infty} = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}+i0}^{\frac{\pi}{2}+i\infty} \tag{86}$$

Integration over one revolution of ϕ [7] yields

$$Z_R = Z_{11} + Z_{12}$$

$$= -2(kb)^4\left(\frac{b}{a}\right)^2 \int_0^{\frac{\pi}{2}+i\infty} \sum_{m=0}^{M}\sum_{j=0}^{M} \tau_m \frac{J_{m+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{m+3/2}} \tau_j \frac{J_{j+3/2}(kb\sin\vartheta)}{(kb\sin\vartheta)^{j+3/2}} \left[1 + J_0(kd\sin\vartheta)\right]\cos^2\vartheta\sin\vartheta\, d\vartheta \tag{87}$$

6.3. Self Impedance

The first term of the integral in (87), which is independent of the spacing *d*, can be shown to be the *self* radiation impedance and gives exactly the same result as the expression for *D*(0) of (70). It differs from the radiation impedance calculated in equations (61) through (65) in that the limit of the radial integral is *b* instead of *a*. Hence we shall write the normalized self resistance as

$$R_{11} = -kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\Re e(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})} \tag{88}$$

and the normalized self reactance as

$$X_{11} = -kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M} \frac{\Im m(\tau_m)}{2^{m+\frac{1}{2}}\Gamma(m+\frac{5}{2})} \tag{89}$$

6.4. Real Mutual Impedance

The second term of the integral in (87), which includes the J_0 function, is the *mutual* impedance The real part can be written

$$R_{12} = -2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M} \frac{\tau_m\tau_j}{(kb)^{m+j}} \int_0^{\frac{\pi}{2}} \frac{J_{m+3/2}(kb\sin\vartheta)}{\sin^{m+3/2}\vartheta} \cdot \frac{J_{j+3/2}(kb\sin\vartheta)}{\sin^{j+3/2}\vartheta} \cdot J_0(kd\sin\vartheta)\cos^2\vartheta\sin\vartheta\, d\vartheta \tag{90}$$

After applying the Lommel expansion (41) to the first and second terms, we have

$$R_{12} = -2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \frac{(kb)^{q+r-m-j}\tau_m\tau_j}{2^{q+r}q!r!} J_{m+q+\frac{3}{2}}(kb) J_{j+r+\frac{3}{2}}(kb) \int_0^{\frac{\pi}{2}} J_0(kd\sin\vartheta)\cos^{2(q+r+1)}\vartheta\sin\vartheta\, d\vartheta \tag{91}$$

which can be solved using Sonine's integral (44) so that the real mutual impedance can finally be written as

$$R_{12} = -2\sqrt{2}\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \frac{\tau_m\tau_j\Gamma(q+r+3/2)}{q!r!(kb)^{m+j+1/2}} \left(\frac{b}{d}\right)^{q+r+3/2} J_{m+q+\frac{3}{2}}(kb) J_{j+r+\frac{3}{2}}(kb) J_{q+r+\frac{3}{2}}(kd) \tag{92}$$

6.5. Imaginary Mutual Impedance

The imaginary part of the second term of (87) can be written

$$X_{12} = -2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M} \frac{\tau_m \tau_j}{(kb)^{m+j}} \int_{\frac{\pi}{2}+i0}^{\frac{\pi}{2}+i\infty} \frac{J_{m+3/2}(kb\sin\vartheta)}{\sin^{m+3/2}\vartheta} \cdot \frac{J_{j+3/2}(kb\sin\vartheta)}{\sin^{j+3/2}\vartheta} \cdot J_0\left(kd\sin\vartheta\right)\cos^2\vartheta\sin\vartheta\, d\vartheta \tag{93}$$

If we now substitute $\vartheta = it + \pi/2$ we obtain an integral with real limits

$$X_{12} = 2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M} \frac{\tau_m \tau_j}{(kb)^{m+j}} \int_0^{\infty} \frac{J_{m+3/2}(kb\cosh t)}{\cosh^{m+3/2} t} \cdot \frac{J_{j+3/2}(kb\cosh t)}{\cosh^{j+3/2} t} \cdot J_0\left(kd\cosh t\right)\sinh^2 t\cosh t\, dt \tag{94}$$

We can apply a Lommel expansion to the first two terms of the integral (94) as follows

$$\frac{J_{m+3/2}(kb\cosh t)}{\cosh^{m+3/2} t} = \frac{J_{m+3/2}\left(kb\sqrt{1+\sinh^2 t}\right)}{(1-\sinh^2 t)^{m/2+3/4}} = \sum_{q=0}^{Q} \frac{(-1)^q (kb)^q \sinh^{2q} t}{2^q q!} J_{m+q+\frac{3}{2}}(kb) \tag{95}$$

Inserting (95) in (94) yields

$$X_{12} = 2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \left\{ \begin{array}{l} \dfrac{(-1)^{q+r}\tau_m\tau_j}{2^{q+r} q!r!(kb)^{q+r-m-j}} J_{m+q+3/2}(kb) J_{j+r+3/2}(kb) \\ \times \displaystyle\int_0^{\infty} J_0\left(kd\cosh t\right)\sinh^{2(q+r+1)} t\cosh t\, dt \end{array} \right\} \tag{96}$$

In order to remove the hyperbolic functions, we shall let $u = \sinh t$ so that

$$X_{12} = 2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \left\{ \begin{array}{l} \dfrac{(-1)^{q+r}\tau_m\tau_j}{2^{q+r} q!r!(kb)^{m+j-q-r}} J_{m+q+3/2}(kb) J_{j+r+3/2}(kb) \\ \times \displaystyle\int_0^{\infty} J_0\left(kd\sqrt{u^2+1}\right) u^{2(q+r+1)}\, du \end{array} \right\} \tag{97}$$

For the Bessel function J_0, we can use the identity [9]

$$J_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\nu\theta} e^{iz\sin\theta} d\theta \tag{98}$$

By substituting $s = ze^{i\theta}/2$ in (98) and integrating along a contour which starts from $-\infty$ on the real axis, encircles the origin once in the positive sense and returns to $-\infty$, we obtain

$$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \oint e^{-i(\nu+1)} \exp\left(s - \frac{z^2}{4s}\right) ds \tag{99}$$

Applying this to (97) with $z = kd\sqrt{u^2+1}$ and $\nu = 0$ yields

$$X_{12} = 2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \left\{ \begin{array}{l} \dfrac{(-1)^{q+r}\tau_m\tau_j(kb)^{q+r-m-j}}{2^{q+r}q!r!} J_{m+q+3/2}(kb)J_{j+r+3/2}(kb) \\ \times\oint\dfrac{1}{s}\exp\left[s-\left(\dfrac{kd}{2}\right)^s\dfrac{1}{s}\right]\int_0^\infty e^{-\left(\frac{kd}{2}\right)^2\frac{u^2}{s}}u^{2(q+r+1)}\,duds \end{array} \right\} \tag{100}$$

After solving the infinite integral we have

$$X_{12} = 2kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \left\{ \begin{array}{l} \dfrac{(-1)^{q+r}\tau_m\tau_j(kb)^{q+r-m-j}}{2^{q+r}q!r!} J_{m+q+3/2}(kb)J_{j+r+3/2}(kb) \\ \times\oint\dfrac{1}{s}\exp\left[s-\left(\dfrac{kd}{2}\right)^2\dfrac{1}{s}\right]\dfrac{\Gamma(q+r+3/2)s^{q+r+3/2}}{2(kd/2)^{2(q+r)+3}}ds \end{array} \right\} \tag{101}$$

where we have used the following identity [9] with $\beta = (kd/2)^2/s$ as follows

$$\int_0^\infty u^{2(n+1)}e^{-\beta u^2}du = \frac{\Gamma(n+3/2)}{2\beta^{n+3/2}} \tag{102}$$

Again, using the identity (99) for the contour integral with $z = kd$ and $\nu = -(q+r+3/2)$, we obtain

$$X_{12} = kb\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \frac{(-1)^{q+r}\tau_m\tau_j\Gamma(q+r+3/2)(kb)^{q+r-m-j}}{2^{q+r}q!r!(kd/2)^{q+r+3/2}} J_{m+q+\frac{3}{2}}(kb)J_{j+r+}\frac{3}{2}(kb)J_{-(q+r+\frac{3}{2})}(kd) \tag{103}$$

Finally, using the identity [9]

$$(-1)^n J_{-(n+3/2)}(kd) = Y_{n+3/2}(kd) \tag{104}$$

we obtain

$$X_{12} = 2\sqrt{2}\left(\frac{b}{a}\right)^2 \sum_{m=0}^{M}\sum_{j=0}^{M}\sum_{q=0}^{Q}\sum_{r=0}^{R} \frac{\tau_m\tau_j\Gamma(q+r+3/2)}{q!r!(kb)^{m+j+1/2}}\left(\frac{b}{d}\right)^{q+r+3/2} J_{m+q+\frac{3}{2}}(kb)J_{j+r+\frac{3}{2}}(kb)Y_{q+r+\frac{3}{2}}(kd) \tag{105}$$

The results of equations (92) and (105) for the mutual radiation resistance and reactance respectively are plotted in Figures 9 and 10 for various ratios of b/a. For $b = \infty$, which represents the infinite baffle case, we have used Pritchard's equations as follows

$$R_{12} = \sqrt{\frac{2}{kd}}\sum_{q=0}^{Q}\sum_{r=0}^{R}\frac{\Gamma(q+r+1)}{q!r!}\left(\frac{a}{d}\right)^{q+r} J_{q+1}(ka)J_{r+1}(ka)J_{q+r+\frac{1}{2}}(kd) \tag{106}$$

$$X_{12} = \sqrt{\frac{2}{kd}}\sum_{q=0}^{Q}\sum_{r=0}^{R}\frac{\Gamma(q+r+1)}{q!r!}\left(\frac{a}{d}\right)^{q+r} J_{q+1}(ka)J_{r+1}(ka)Y_{q+r+\frac{1}{2}}(kd) \tag{107}$$

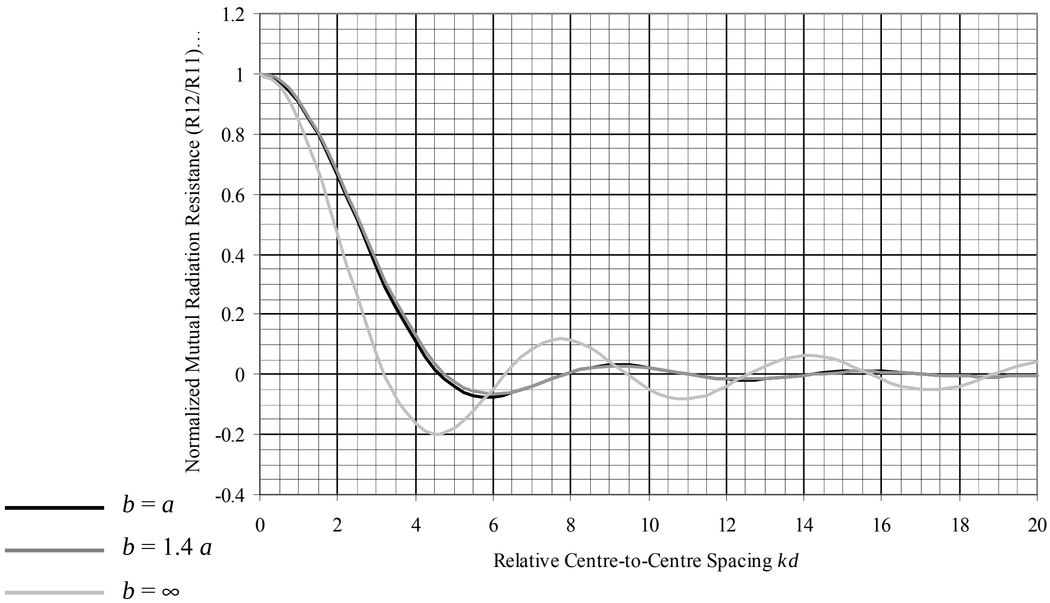

Figure 9. *Mutual Radiation Resistance of a Two Rigid Discs in Open Finite Circular Baffles at ka* = 1

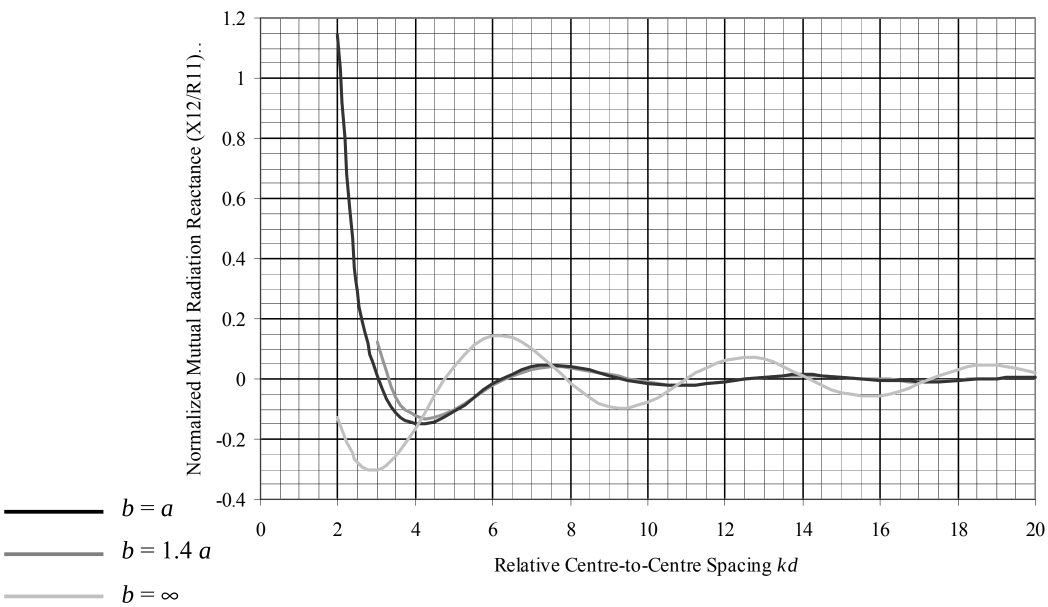

Figure 10. *Mutual Radiation Reactance of a Two Rigid Discs in Open Finite Circular Baffles at ka* = 1

7. MUTUAL RADIATION CHARACTERISTICS OF TWO RIGID DISCS IN CLOSED FINITE CIRCULAR BAFFLES IN FREE SPACE

In order to plot the mutual radiation resistance shown in Figure 11, we simply add equations (94) and (107) and divide the result by 2. Similarly, the mutual radiation reactance shown in Figure 12 is obtained by adding equations (106) and (108) and dividing the result by 2.

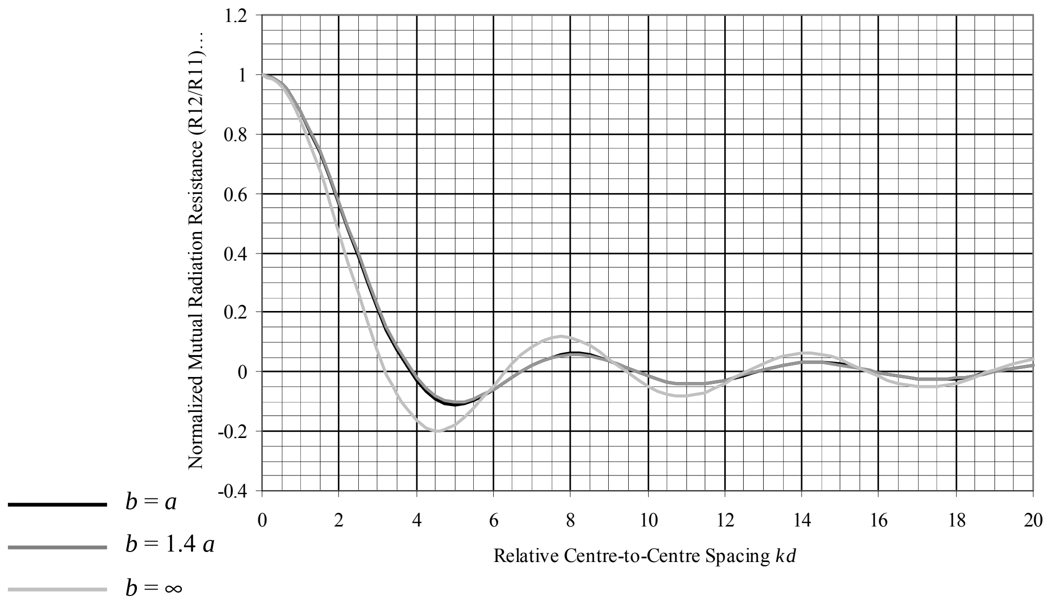

Figure 11. *Mutual Radiation Resistance of Two Rigid Discs in Closed Finite Circular Baffles at ka* = 1

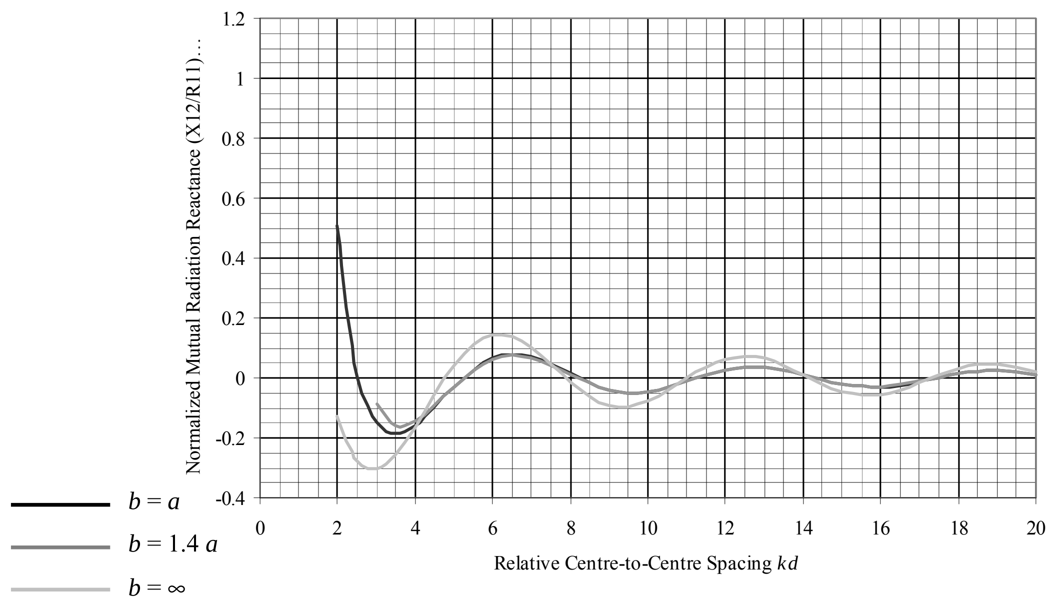

Figure 12. *Mutual Radiation Reactance of Two Rigid Discs in Closed Finite Circular Baffles at ka* = 1

8. CONCLUSIONS

A method for calculating the radiation characteristics of two rigid discs in open or closed finite circular baffles has been described. Furthermore, a simple unified method has been presented for reproducing the results given by Beranek [1] for the radiation characteristics of a single disc which gives good agreement with the originals. Referring to Figure 4, it can be seen that in the case of a rigid disc with no baffle, the on-axis sound output falls at 6dB/octave for small values of ka due to the decreasing path difference (as a proportion of wavelength λ) between the anti-phase rear radiation and the front radiation which it cancels. At larger values of ka, the rear radiation moves in and out of phase with that from the front, but the summation and cancellation effects are fairly small, the largest peak being 3 dB at $ka = \pi/\sqrt{2}$ or $\lambda = 2\sqrt{2}\ a$. The reason for this is that rear radiation comprises many point sources spread all across the radius of the disc, each with a different path length to the front, so that at no particular frequency do they produce a combined source that is either directly in phase or out of phase with the output from the front. However, if we include a circular baffle and increase its size, the actual radiating area decreases in proportion to the total so that it behaves more like a coherent point source. Hence, in the case of $b = 4a$, we see a deep null at $ka = \pi/2$ or $\lambda = 4a$, which is the distance from the centre to the edge. Of course, a disc at the centre of a circular baffle is the "worst case" and it would be interesting to compare these results with those of an offset disc in a circular, rectangular or elliptical baffle, for example, in order to "smear" the path difference effect.

One would expect that if the size of the baffle were increased still further, the response would converge towards that of an infinite baffle, which is a ruler flat line. However, the response becomes ever harder to calculate as we increase the ratio of b/a and the limit of the power series M has to be increased at higher frequencies to reflect the more impulse-like surface pressure distribution. Of course, a larger baffle requires more calculation points P too. This means that we need to extend the precision in order to avoid numerical problems associated with a badly conditioned matrix, especially at low frequencies where R_R is much smaller than X_R. In this paper, P = 30 was used for $b = a$ and P = 120 for $b = 4a$. M was varied from 4 at low frequencies without a baffle up to 10 at higher frequencies with $b = 4a$. As for the other function series limits, $Q = R = 20$ was found to be sufficient for all instances.

According to the product theorem (84), the response in the xz plane ($\phi = 0$) is exactly the same as that for a single disc except for a factor of two. This result is perhaps slightly surprising considering that we have a case of mutual scattering in addition to mutual impedance. In the case of an infinite baffle there is simply mutual impedance because when one radiator is "switched off", the other just "sees" an undisturbed infinite plane. However, this result can be verified using FEM/BEM analysis and holds even if the baffles (or unbaffled discs) are just about touching.

As Pritchard pointed out in the case of two discs in an infinite baffle, the fact the power series for the mutual reactance does not converge for $d < 2b$ is merely academic because beyond this point the discs (or finite baffles) are partially coalesced. It is interesting to note that in the case of close proximity, the mutual reactance is higher for two unbaffled discs than for two discs in an infinite baffle. However, beyond this point, the mutual reactance of the unbaffled discs decays far more rapidly. This is possibly due to the fact that the discs are located within the "nulls" of each other's dipole directional responses. The same trend can be seen for the mutual resistance except that the region for which the unbaffled impedance is higher is extended slightly. The curves for $b = a$ and $b = 1.4a$ are almost coincident across the range.

As one would expect from Babinet's principle, all of the results for discs in closed finite baffles lie somewhere between those for the corresponding discs in open finite baffles and those for discs in an infinite baffle.

As one final observation, Morse and Ingard [5] present a partial derivation for a disc in free space based upon the assumption of constant pressure across the surface, as in the Kirchhoff theory of diffraction. If the expression for the velocity is integrated over the surface to give the volume velocity and divided by the total driving force per unit area, this yields an integral for the normalized real radiation admittance (or conductance), the solution to which is

$$G_{\mathrm{R}} = 1 + \frac{1}{ka} J_1(2ka) - 2J_0(2ka) - \pi\left[J_1(2ka)\mathbf{H}_0(2ka) - J_0(2ka)\mathbf{H}_1(2ka)\right] \quad (108)$$

This is quite a good approximation to the rigorous solution using Streng's method except that the ripples are missing. Unfortunately, the integral for the imaginary radiation admittance (or susceptance) B_{R} has a non converging integrand and therefore cannot be solved. However, a reasonable approximation can be made by taking the imaginary radiation admittance of a disc in an infinite baffle and doubling it.

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