Acoustic modeling using the ultra weak variational formulation

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1 – Acoustic fields

- **The aim is to simulate high frequency pressure fields in complex geometries (also in inhomogeneous media)**
- **Wide range of applications (ultrasonics, audio acoustics: e.g. HRTF)**
- **Linear acoustic model in frequency domain** $P(r,t) = p(r)\exp(-i\omega t)$

$$\nabla \cdot \left(\frac{1}{\rho}\nabla P\right) - \frac{1}{\rho c^2}\frac{\partial^2 P}{\partial t^2} = 0 \quad \Rightarrow \quad \nabla \cdot (\frac{1}{\rho}\nabla p) + \frac{\kappa^2}{\rho} p = 0$$

where p **= acoustic pressure,** ρ **= density,** c **= speed of sound,** $\omega = 2\pi f$ **angular frequency and** $\kappa = \omega / c$ **= wave number**

2 – Numerical methods for high frequency problems

1. **Ray-approximations:**
   - **Fast in simple geometries**
   - **Insufficient for complex media**
2. **Finite difference methods (FD, FDTD)**
   - **Widely used for time-domain problems**
3. **Boundary element methods (BEM)**
   - **Difficult for inhomogeneous media**
4. **Finite element methods (FEM)**
   - **Flexible for general geometries**

Methods 2.-4. require a dense spatial discretization.

3 – Drawback of standard FEM

- The basis functions are typically low-order polynomials.
- A rule of thumb: 10 elements / wavelength
- Numerical pollution:

  An error estimate for a low-order FEM:

$$\frac{|p - p_h|_1}{|p|_1} < C_1 \kappa h + C_2 \kappa^3 h^2,$$

where $|\cdot|_1$ is $H^1(\Omega)$-seminorm, h = elements size, C_1 and C_2 are constants.

4 – Methods that use “wave-like” basis functions

- Infinite elements (Bettess and Zienkiewicz 1977)
- Partition of unity finite element method = PUM (Babuška and Melenk 1997)
- Least squares method (Monk and Wang 1999)
- Discontinuous enrichment method (Farhat et al. 2001)
- Discontinuous Galerkin method (Farhat et al. 2003)
- Plane wave basis in integral equations (Perrey-Debain et al. 2002)
- Ultra weak variational formulation (Després 1994, Cessenat and Després 1998)

5 – Ultra weak variational formulation (UWVF)

- **Uses standard finite element meshes**
- **A new variational formulation on element boundaries**
- **In stead of solving the pressure field directly, we solve a new function on element interfaces**

$$\chi = \Big(- \frac{1}{\rho} \frac{\partial p}{\partial n} - i\varsigma p \Big)$$

- **Solution in each element is approximated using plane wave basis functions $\Rightarrow$ allows larger elements than FEM $\Rightarrow$ Reduced computational burden**
- **Simulation are still demanding : We use a parallel code on a PC cluster consisting of 24 Pentium 4 with 48 GB total RAM.**

6 – Just to show the UWVF

$$\sum_{k=1}^{N}\int_{\partial K_k}\frac{1}{\varsigma}\chi_k\overline{\left(-\frac{1}{\rho_k}\frac{\partial}{\partial n_k}-i\varsigma\right)v_k}-\sum_{k=1}^{N}\sum_{j=1}^{N}\int_{\Sigma_{k,j}}\frac{1}{\varsigma}\chi_j\overline{\left(\frac{1}{\rho_k}\frac{\partial}{\partial n_k}-i\varsigma\right)v_k}$$
$$-\sum_{k=1}^{N}\int_{\Gamma_k}\frac{Q}{\varsigma}\chi_k\overline{\left(\frac{1}{\rho_k}\frac{\partial}{\partial n_k}-i\varsigma\right)v_k}=\sum_{k=1}^{N}\int_{\Gamma_k}\frac{1}{\varsigma}g\overline{\left(\frac{1}{\rho_k}\frac{\partial}{\partial n_k}-i\varsigma\right)v_k}, \quad (1)$$

for all piecewise smooth functions v_k satisfying the adjoint Helmholtz equation
$\nabla\cdot(A_k\nabla\overline{v}_k)+\kappa_k^2\eta^2\overline{v}_k=0$ in K_k. Where

K_k = a finite element

$\Sigma_{k,j}$ = an interface between elements K_k and K_j

Γ_k = an element face on the external boundary.

7 – Discrete UWVF

- **The basis functions are complex conjugated plane waves:**

$$\varphi_{k,\ell} = \begin{cases} \exp(i\overline{\kappa}_k d_{k,\ell} \cdot r) & \textbf{in } K_k \\ 0 & \textbf{elsewhere,} \end{cases}$$

- **The discrete problems is written as the sparse matrix equation**

$$(I - D^{-1}C)X = D^{-1}b,$$

where D and C are sparse block matrices and X includes weights for basis functions for each element. The matrix equation is solved using the stabilized bi-conjugate gradient.

8 – Our contribution to UWVF

- Stabilization of the method by using element-wise varying number of basis functions.
- Extension of the UWVF method for elastic wave problems (currently in 2D).
- Improved truncation of unbounded problems (= Perfectly matched layer (PML))
- Verification of the UWVF-Helmholtz model for ultrasound problems

9 – A test problem: Scattering from a sphere with $\phi = 20$ cm

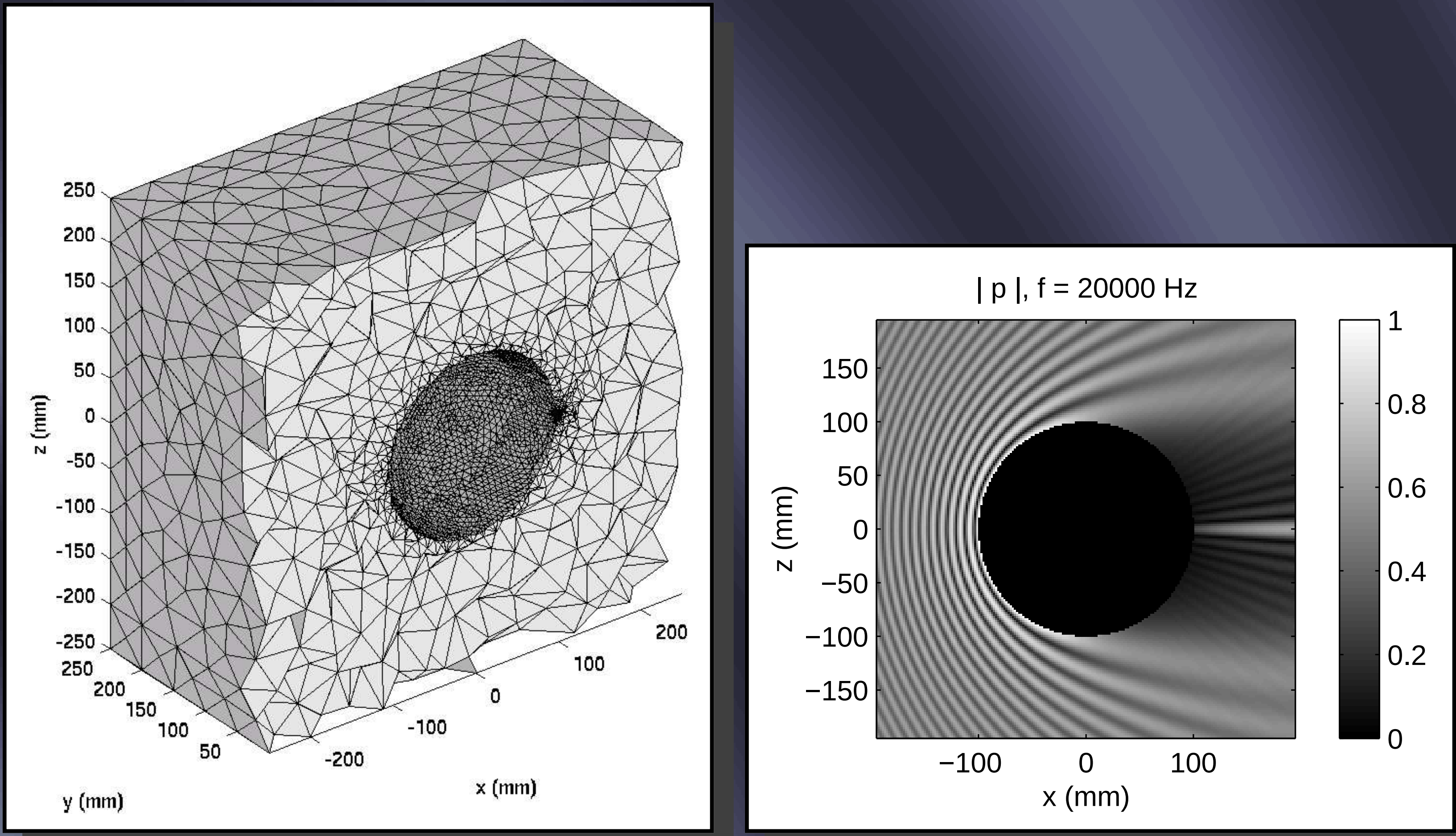


Figure 1: *Left:* **The mesh.** *Right:* **Solution for a plane wave propagating from left to right at 20 kHz.**

10 – Accuracy of the UWVF approximation

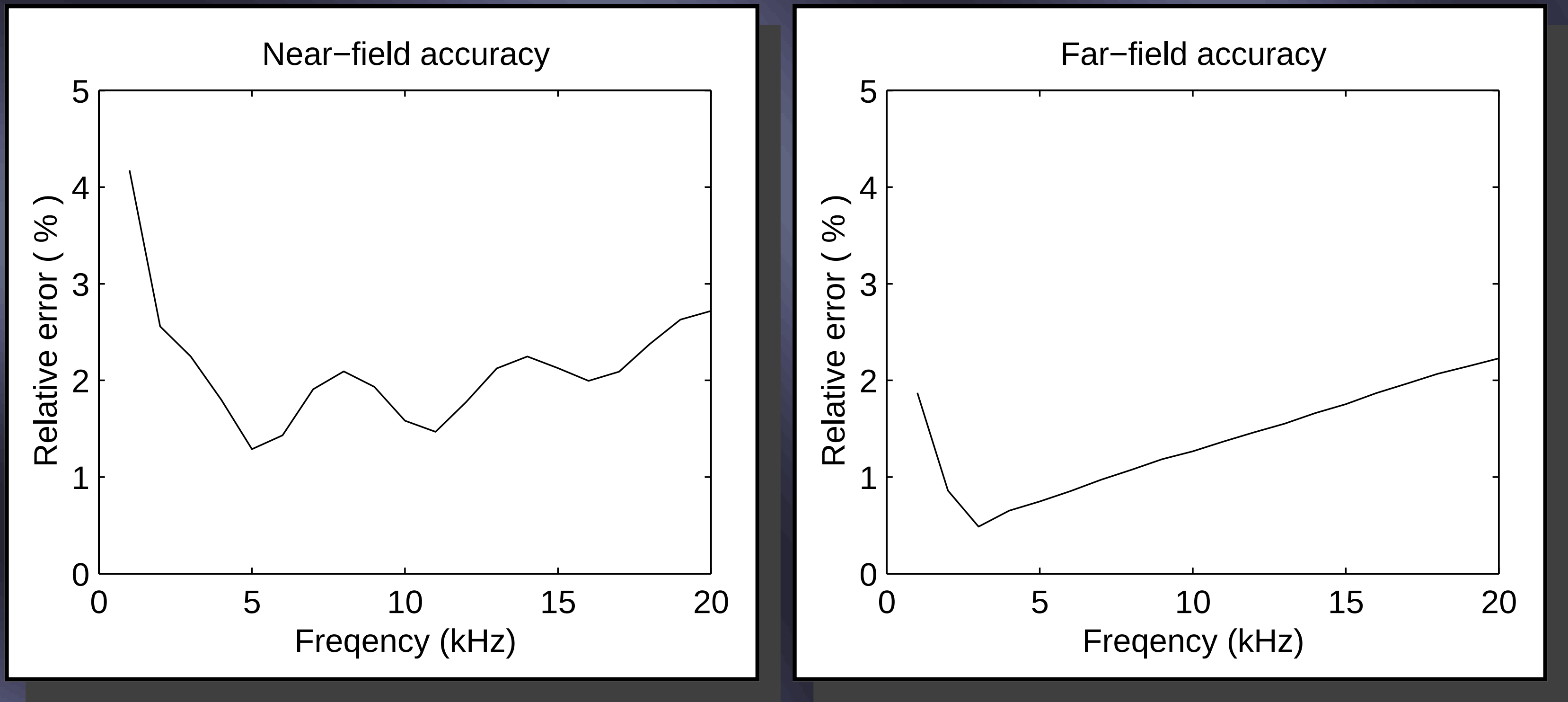
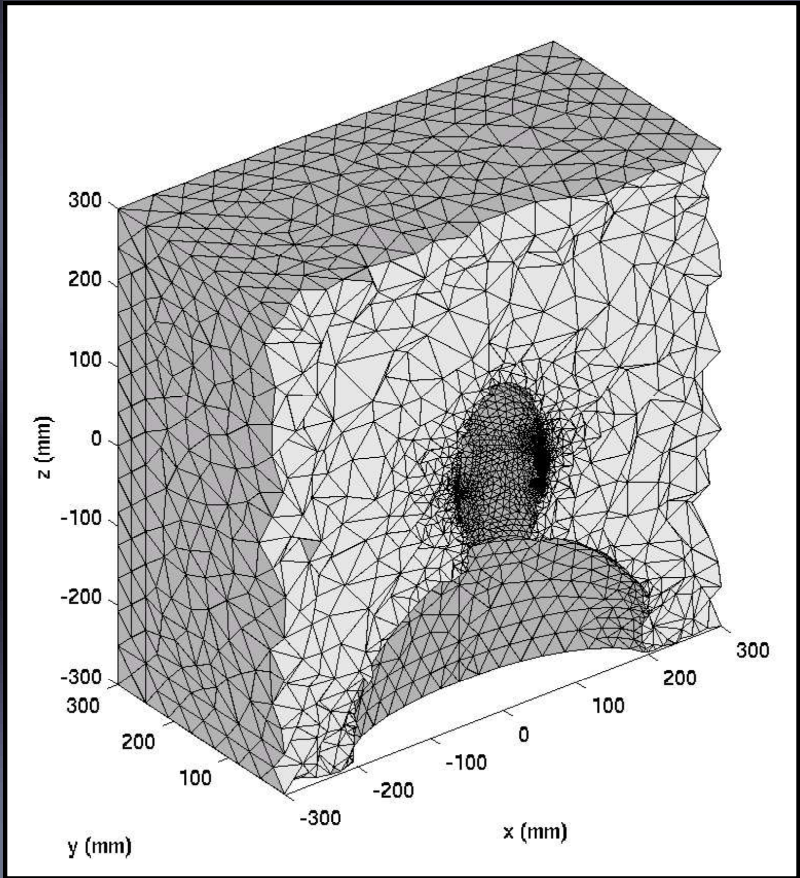
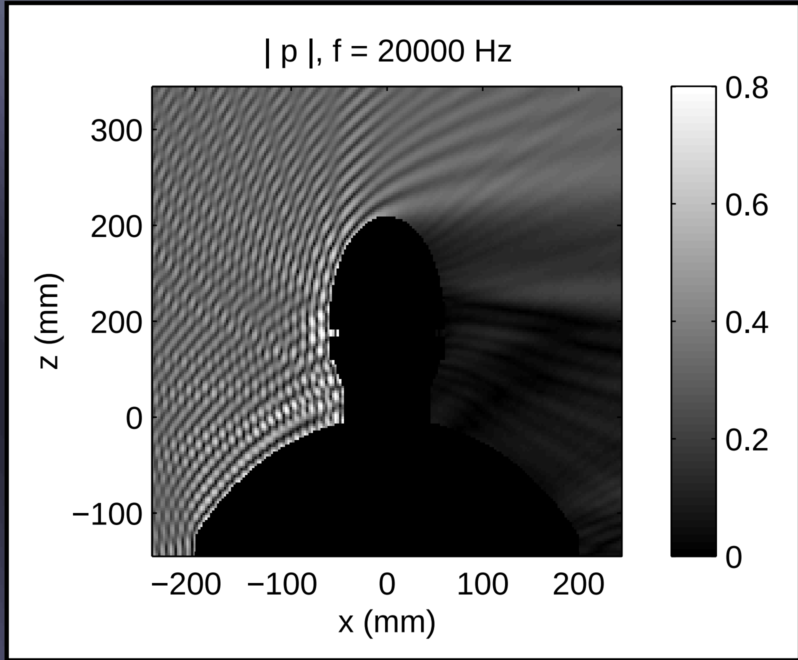


Figure 2: The relative error of the UWVF approximation in near- and far- field. All results are computed in the same mesh by varying the number of basis functions.

11 – Field surrounding a human figure





12 – Time as a function of frequency

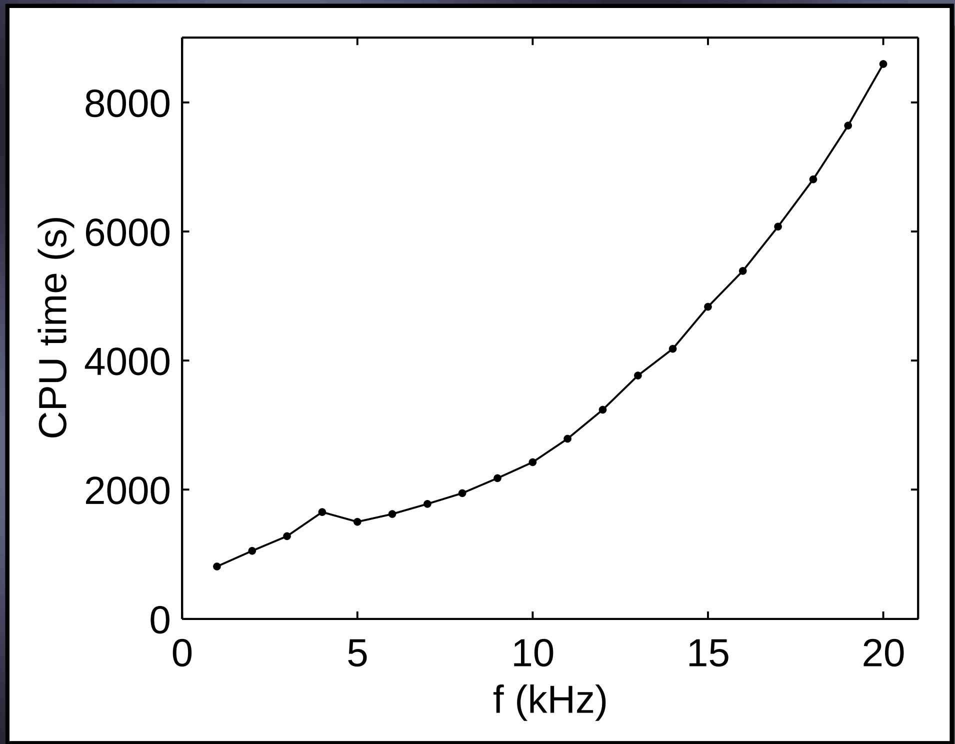
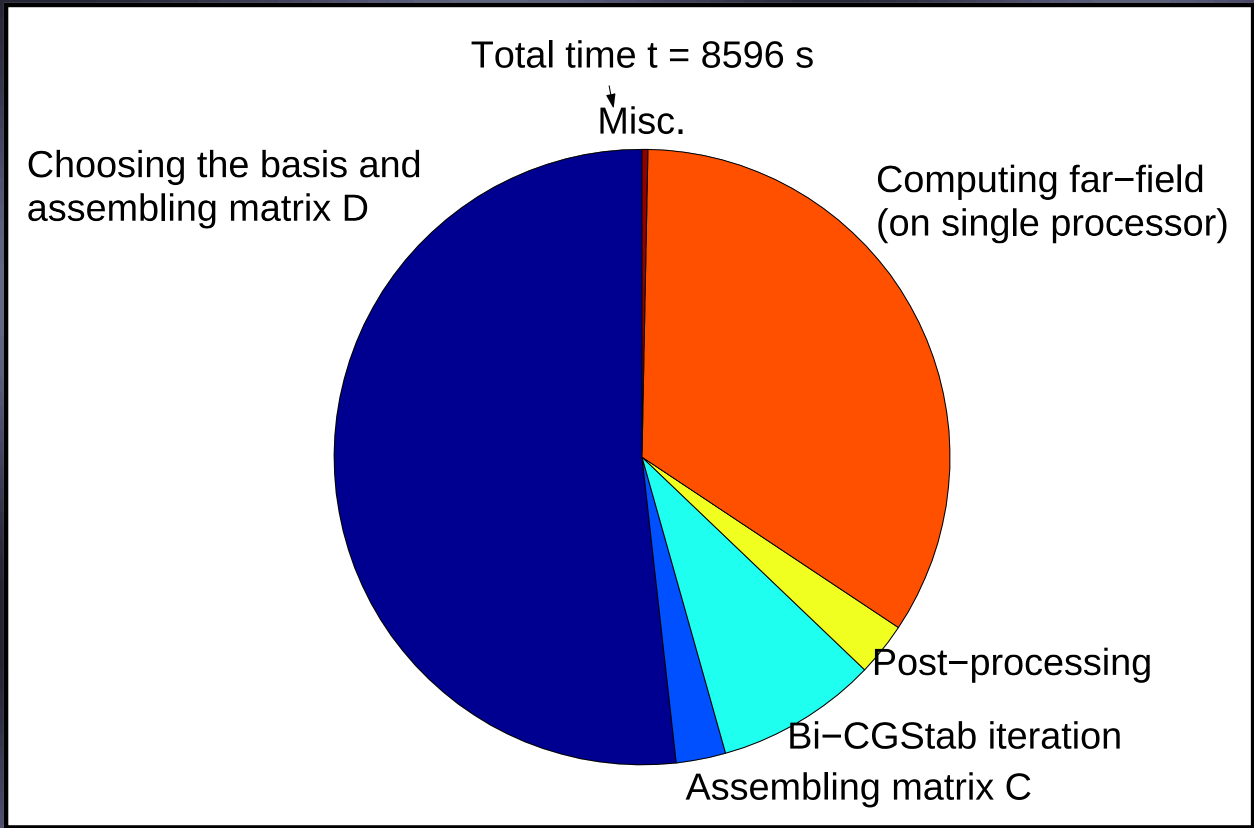


Figure 3: All frequencies are computed in the same mesh by varying the number of basis functions and by using 24 processors.

13 – Time spent for different sub-procedures



14 – Conclusions

- The UWVF is a promising alternative for the standard FEM.
- A possible advantage over BEM: The UWVF is well suited for problems in inhomogeneous media.
- A faster method for choosing the right number of basis functions is needed.
- Extension of the method for coupled fluid-solid problems.